# NOTE ON A UNIQUENESS RESULT FOR THE NONLINEAR DIFFERENTIAL EQUATION WITH TIME-SINGULAR COEFFICENTS

**Nguyen Minh Dien**[(1)]

*(1) Thu Dau Mot University*
*Corresponding author: diennm@tdmu.edu.vn*

**DOI: 10.37550/tdmu.EJS/2024.02.546**

**Article Info**
**Volume:** 6
**Issue:** 02
**June** 2024
**Received:** Jan. 17th, 2024
**Accepted:** April 8th, 2024
**Page No:** 202-206

**Abstract**
In this note, under some new appropriate conditions, we obtain a Nagumo-type uniqueness result for a nonlinear differential equation. We emphasize that our result still work with the source functions having time-singular coefficients. An example is given to illustrate the theoretical result.

**Keywords:** Initial value conditions; Nagumo-type source; Nagumo-type unique

## 1 Introduction

In the pioneering work of Nagumo (Nagumo, 1926), the author showed that the following problem

$$\begin{cases} y' = f(y,t), \\ y(0) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $\lim\limits_{t\to 0} f(y,t) = 0$ uniformly for $|y| \leq 1$, has a unique trivial solution, provided that

$$|f(y,t) - f(z,t)| \leq \frac{1}{t}|y - z|$$

for $t \in [0,1]$ and $y, z \in \mathbb{R}$. After that, there are various papers investigated the Nagumo-type unique results for both ordinary differential equations and fractional differential equations (see Athanassov, 1990; Constantin, 2010; Delbosco and Rodino, 1996; Dien, 21; Dien 22).

In this note, we intend to improve some conditions in the works of Athanassov (Athanassov, 1990) and Constantin (Constantin, 2010). To do this, we consider the following differential equation

$$y' = f(t,y) \tag{1.1}$$

together with the initial condition

$$y(0) = 0, \tag{1.2}$$

where $f$ is a continuous function on $\mathcal{D} = \{(y, t) \in \mathbb{R}^2 : \ |y| \leq 1, \ 0 < t \leq 1\}$. We would like to propose some new sufficient conditions such that the initial problem (1.1) and (1.2) has a unique solution.

The rest of this note is organized as follows. In section 2, we present the main result of the note. In section 3, we provide an example to illustrate the theoretical result.

## 2 Main result

This section is devoted to stating and proving the main result of this note. Let us begin this section by giving a primary lemma which we will use to prove the main result.

**Lemma 2.1.** *If $y \in C^1[0, 1]$ and $y(0) = 0$ then there exists a positive number $M$ such that*

$$|y(t)| \leq Mt.$$

Proof. Let us put $M = \max_{0 \leq t \leq 1} |y'(t)|$. We have

$$\begin{aligned} |y(t)| &= \left| \int_0^t y'(s) \, \mathrm{d}s \right| \\ &\leq \int_0^t |y'(s)| \, \mathrm{d}s \\ &\leq M \int_0^t \mathrm{d}s = Mt. \end{aligned}$$

This completes the proof of Lemma. □

We continue with the following definition:

**Definition 2.2.** *The class of functions $\mathfrak{F}$ is the set of all functions $\omega$ satisfying:*

*(i). $\omega$ is a strictly increasing function on $[0, 1]$*

*(ii). $\int_0^r \frac{\omega(s)}{s} \, \mathrm{d}s \leq r$ for any $r \in (0, 1)$.*

Based on the above definition, we state the main result of this note as follows.

**Theorem 2.3.** *Let $u \in C^1[0, 1]$ such that $\lim_{t \to 0^+} u(t) = 0$ and $u'(t) > 0$ for any $t \in (0, 1)$. Suppose that*

$$\lim_{t \to 0^+} \frac{f(xt, t)}{u'(t)} = 0 \tag{2.1}$$

*uniformly with respect to* $|x| \le M$ *for each* $M > 0$. *Moreover, there exists a function* $\omega \in \mathfrak{F}$ *such that*

$$|f(y,t)| \le \frac{u'(t)}{u(t)}\omega(|y|) \tag{2.2}$$

*for any* $t \in (0,1]$. *Then the problem (1.1) and (1.2) has only the trivial solution in* $C^1[0,1]$.

**Remark 2.4.** *The condition (2.1) is weaker than the condition* $\lim_{t\to 0^+} \frac{f(y,t)}{u'(t)} = 0$ *uniformly with respect to* $|y| \le 1$ *in the work of Constantin (Constantin, 2010).*

Proof. We prove by contradiction, assuming on the contrary that the problem (1.1) and (1.2) has a nontrivial solution $y \in C^1[0,1]$. Using L'Hospital's rule and (2.1) together with Lemma 2.1, we have

$$\begin{aligned}\lim_{t\to 0}\frac{y(t)}{u(t)} &= \lim_{t\to 0}\frac{y'(t)}{u'(t)}\\ &= \lim_{t\to 0}\frac{f(y(t),t)}{u'(t)}\\ &= 0.\end{aligned}$$

We define a continuous function on $[0,1]$ for some $t_0 > 0$ as follows

$$\zeta(t) = \begin{cases} 0 & \text{as } t = 0,\\ \dfrac{y(t)}{u(t)} & \text{as } 0 < t \le 1.\end{cases}$$

Since $\zeta \in C[0,t_0]$, so there exists $t^* \in [0,t_0]$ such that $\zeta(t^*) = \max_{0\le t\le t_0} |\zeta(t)| > 0$. Then, we have

$$\begin{aligned}|y(t^*)| &\le \int_0^{t^*} |f(y(s),s)|\,\mathrm{d}s\\ &\le \int_0^{t^*} \frac{u'(s)}{u(s)}\omega(|y(s)|)\,\mathrm{d}s\\ &= \int_0^{t^*} \frac{u'(s)}{u(s)}\omega\left(\frac{|y(s)|}{u(s)}u(s)\right)\,\mathrm{d}s\\ &< \int_0^{t^*} \frac{u'(s)}{u(s)}\omega(\zeta(t^*)u(s))\,\mathrm{d}s\\ &= \int_0^{\zeta(t^*)u(t^*)} \frac{\omega(z)}{z}\,\mathrm{d}z\\ &\le \zeta(t^*)u(t^*).\end{aligned}$$

This leads to

$$\zeta(t^*) = \frac{|y(t^*)|}{u(t^*)} < \zeta(t^*),$$

which is absurd. So, the proof of the theorem is completed. □

# 3 Example

In this part, we provide an example to show the applicability of the obtained result. This example also shows that the improvement of our result compared with some previous one.

**Example 3.1.** *Find all solutions of the following problem*

$$\begin{cases} y' = \dfrac{y\cos t}{\sin t(\ln t - 1)}, & (0 < t \le 1), \\ y(0) = 0. \end{cases} \tag{3.1}$$

*in* $C^1[0,1]$.

**Solve.** Remark that $y(t) \equiv 0$ is a solution of the problem (3.1). We will prove that the problem has nontrivial solution. In fact, put $u(t) = \sin t$, then one has

$$\begin{aligned} f(y,t) &= \frac{y\cos t}{\sin t(\ln t - 1)} \\ &= \frac{u'(t)}{u(t)(\ln t - 1)} y. \end{aligned}$$

Since $|\ln t - 1| \ge 1$ for any $t \in (0,1]$, this gives

$$\begin{aligned} |f(y,t)| &\le \frac{u'(t)}{u(t)}|y| \\ &= \frac{u'(t)}{u(t)}\omega(|y|), \end{aligned} \tag{3.2}$$

where $\omega(z) = z$. It is obvious that $\omega \in \mathfrak{F}$. Moreover, we also have

$$\begin{aligned} \lim_{t\to 0^+} f(xt,t) &= \lim_{t\to 0^+} \frac{xt\cos t}{\sin t(\ln t - 1)} \\ &= \lim_{t\to 0^+} \frac{t\cos t}{\sin t} \lim_{t\to 0^+} \frac{x}{\ln t - 1} \\ &= 0 \end{aligned} \tag{3.3}$$

uniformly with respect to $|x| \le M$ for some $M > 0$. From (3.2) and (3.3), we find that all the assumptions of Theorem 2.3 are true. Therefore, we conclude that the problem (3.1) has a unique solution $y(t) = 0$. □

**Remark 3.1.** *Note that*

$$\lim_{t\to 0^+} f(y,t) = \lim_{t\to 0^+} \frac{y\cos t}{\sin t(\ln t - 1)} \not\to 0$$

*uniformly with respect to* $|y| \le 1$*, so, we can not use the result in (Athanassov, 1990) or (Constantin, 2010) to conclude the problem (3.1) has only a trivial solution. This shows the improvement of our result if compared with some previous results.*

## References

Athanassov, Z. S. (1990). *Uniqueness and convergence of successive approximations for ordinary differential equations*. Math. Japon, *35*(2), 351-367.

Constantin, A. (2010). *On Nagumo's theorem*. Proc. Japan Acad., 86, 41-44.

Delbosco, D., Rodino, L. (1996). *Existence and uniqueness for a nonlinear fractional differential equation*. J. Math. Anal. Appl., 204, 609-625.

Dien, N. M. (2021). *Existence and continuity results for a nonlinear fractional Langevin equation with a weakly singular source*. J. Integral Equ. Appl., *33*(3): 349-369.

Dien, N. M. (2022). *On mild solutions of the generalized nonlinear fractional pseudoparabolic equation with a nonlocal condition*. Fract. Calc. Appl. Anal., *25*(2), 559-83. DOI: 10.1007/s13540-022-00024-4.

Nagumo, M. (1926). *Eine hinreichende Bedingung für die Unit ät der Löosung von Differentialgleichungen erster Ordnung*. Japan. J. Math., 3, 107-112.