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The nice *m*-system of parameters for Artinian modules

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ABSTRACT

This paper restates the definition of the nice m-system of parameters for Artinian modules. It also shows its effects on the differences between lengths and multiplicities of certain systems of parameters for Artinian modules:

$$I\left(\underline{x}(\underline{n});A\right)=\ell_R\left(0:_A\left(x_1^{n_1},x_2^{n_2},...,x_d^{n_d}\right)R\right)-e\left(x_1^{n_1},x_2^{n_2},...,x_d^{n_d};A\right)$$

In particular, if $\underline{x}$ *is a nice m-system of parameters then the function* $I\left(\underline{x}(\underline{n});A\right)$ *is a polynomial having very nice form. Moreover, we will prove some properties of the nice m-system of parameters for Artinian modules. Especially, its effect on the annihilation of local homology modules of Artinian module A.*

Keywords: *annihilation, Artinian module, function of certain systems of parameters, local homology, nice m-system of parameters*

1. Introduction

Throughout this paper, (*R*, *m*) is a commutative Noetherian local ring with the maximal ideal *m*, and *A* is an Artinian *R*-module with N-dim*A* = *d*.

+ The *R*-module $\varprojlim_t Tor_i^R\left(R/m^t;A\right)$ is called ith-local homology module of *A* with respect to m and denoted by $H_i^m(A)$.

+ Let $\underline{n}=(n_1,n_2,...,n_d)$ be a d-tuple of positive integers. For each system of parameters (s.o.p) $\underline{x}=(x_1,x_2,...,x_d)$ of A, we consider

$$I(\underline{x}(\underline{n});A)=\ell_R\left(0:_A(x_1^{n_1},x_2^{n_2},...,x_d^{n_d})R\right)-e\left(x_1^{n_1},x_2^{n_2},...,x_d^{n_d};A\right)$$

as a function d-variables on $n_1,n_2,...,n_d$.

Let $I(A)=sup_x\, I(\underline{x};A)$ where $\underline{x}$ runs over all s.o.p of A.

The value of function $I(\underline{x};A)$ and the annihilation of local homology modules of A help us classify many different types of modules. Moreover, they also give us lots of information about different types of modules (see [3]). Such as:

+ $I(A)=0$: A is a co-Cohen-Macaulay module.

+ $I(A)<\infty$: A is a Generalized co-Cohen-Macaulay module.

+ $I(\underline{x};A)$ is a constant for all s.o.p of A: A is a co-Buchsbaum module.

+ If A is a Generalized co-Cohen-Macaulay module, there exists an m-primary ideal q such that $qH_i^m(A)=0$ for all i = 1, …, d – 1.

+ If A is a co-Buchsbaum module, $mH_i^m(A)=0$ for all i = 1, …, d – 1.

However, $I(\underline{x}(\underline{n});A)$ may be not a polynomial on $n_1,n_2,...,n_d$ even when $n_1,n_2,...,n_d$ large enough (see [1]), but [2] has shown that if $\underline{x}$ is a nice m-systems of parameters, $I(\underline{x}(\underline{n});A)$ is a polynomial with simple form. In addition, a nice s.o.p of A also annihilates local homology modules of A. Thus, in this paper we will restate the definition of the nice m-s.o.p for Artinian modules, the effect of the nice m-s.o.p on the calculation formula of function $I(\underline{x}(\underline{n});A)$ and continue studying some its properties. Especially its effect on the annihilation of local homology modules of A.

2. Preliminaries

Lemma 2.1([1]). Assume $(Ann_R A)\hat{R}=Ann_{\hat{R}}A$ *and* $\underline{x}=(x_1,x_2,...,x_d)$ *is a s.o.p of Artinian R-module A. Then, there exits* $j\in\{1,2,...,d\}$ *such that* x_j *is a pseudo-A-coregular element.*

Lemma 2.2 ([3]). Let $x \in R$ *be a pseudo-A-coregular element. Then* $\ell_R\left(A/xA\right) < \infty$.

Lemma 2.3 ([4]). Let M be an R-module, I be an ideal of R. Then for all i ≥ 0,

$$\bigcap_{s>0} I^s H_i^I(M) = 0.$$

Lemma 2.4 ([1]). Let s a positive integer such that $m^t A = m^s A, \forall t \geq s.$ *Then*

$$H_0^m(A) = A/m^s A.$$

Lemma 2.5. ([3]). For every s.o.p $\underline{x}$ *of A, we have*

$$\ell_R\left(0:_A \underline{x}R\right) - e\left(\underline{x}; A\right) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_R\left(H_i^m(A)\right).$$

Moreover, if $\ell_R\left(H_i^m(A)\right) < \infty$ *for all i < d, then there exists an m-primary ideal q such that the equality holds for every s.o.p* $\underline{x}$ *contained in q.*

Definition 2.6 ([2]).

** The sequence* $x_1,...,x_t \in m$ *is called an m-sequence for A if:*

(i) $x_k \notin \sum_{s\neq k} x_s R$ *for all k = 1, …, t,*

(ii) $x_k\left(0:_A \left(x_1,...,x_{i-1}\right)R\right) = x_k x_i \left(0:_A \left(x_1,...,x_{i-1}\right)R\right)$ *for all* $1 \leq i \leq k \leq t \left(x_0 = 0\right)$.

** The sequence* $x_1,...,x_t \in m$ *is called a strong m-sequence for A if* $x_1^{n_1},...,x_t^{n_t}$ *is m-sequence for all* $\left(n_1,...,n_t\right) \in \mathbb{N}^t$.

** A strong m-sequence* $x_1,...,x_t \in m$ *is called a nice m-sequence for A if:*

(i) t = 1; or

(ii) t > 1 and $x_1,...,x_{i-1}$ *is a strong m-sequence of* $0:_A \left(x_i^{n_i},...,x_t^{n_t}\right)R$ *for all* $2 \leq i \leq t$ *and for all* $n_i,...,n_t \in \mathbb{N}$.

** A s.o.p for A is called a nice m-s.o.p if it is a nice m-sequence.*

Lemma 2.7 ([2]). Let $x_1,...,x_t$ *be an m-sequence for A. Then:*

(i) $x_i\left(0:_A \left(x_1,...,x_{i-1}\right)R\right) = x_i^n \left(0:_A \left(x_1,...,x_{i-1}\right)R\right)$ *for all* $1 \leq i \leq t$ *and* $n \in \mathbb{N}$;

(ii) for every (i, k) with $1 \leq i \leq k \leq t$ *we have*

$$x_k\left(0:_A\left(x_1,...,x_{i-1}\right)R\right)\subseteq x_i\left(0:_A\left(x_1,...,x_{i-1}\right)R\right);$$

(iii) $x_2,...,x_t$ *is an m-sequence for* $0:_A x_1$.

The following theorem shows that $I\left(\underline{x}\left(\underline{n}\right);A\right)$ will be a polynomial when $\underline{x}=\left(x_1,x_2,...,x_d\right)$ is a nice *m*-s.o.p for *A*. Furthermore, in this case it has a nice form.

Theorem 2.8 ([2]). Let $\underline{x}=\left(x_1,x_2,...,x_d\right)$ *be a s.o.p for A. Then the following three conditions are equivalent:*

(i) $\underline{x}$ *is a nice m-s.o.p for A;*

(ii) there exist non-negative intergers $\alpha_0\left(\underline{x},A\right),...,\alpha_{d-1}\left(\underline{x},A\right)$ *such that*

$$I\left(\underline{x}\left(\underline{n}\right);A\right)=\alpha_0\left(\underline{x},A\right)+\sum_{i=1}^{d-1}n_1...n_i.\alpha_i\left(\underline{x},A\right)$$

for all $n_1,...,n_d\geq 1$;

(iii)

$$I\left(\underline{x}\left(\underline{n}\right);A\right)=\ell_R\left(\frac{0:_A\left(x_2,...,x_d\right)R}{x_1\left(0:_A\left(x_2,...,x_d\right)R\right)}\right)+\sum_{i=1}^{d-1}n_1...n_i.e\left(x_1,...,x_i;\frac{0:_A\left(x_{i+2},...,x_d\right)R}{x_{i+1}\left(0:_A\left(x_{i+2},...,x_d\right)R\right)}\right)$$

for all $n_1,...,n_d\geq 1$.

3. Main results

In this section, we give some corollaries of Theorem 2.8.

Corollary 3.1. Let $x_1,x_2,...,x_d$ *be a nice m-s.o.p for A with N-dimA =* $d\geq 2$. *Then*

i) *For all* $n_1,...,n_d\in\mathbb{N}$ *we have*

$$I\left(x_1^{n_1},x_2^{n_2},...,x_d^{n_d};A\right)=I\left(x_1^{n_1},x_2^{n_2},...,x_d;A\right).$$

ii) For all $n_2,...,n_d\in\mathbb{N}$ *we have*

$$I\left(x_2^{n_2},...,x_d^{n_d};0:_A x_1\right)=I\left(x_1,x_2^{n_2},...,x_d^{n_d};A\right).$$

Proof.

i) From (iii) of Theorem 2.8, we find that $I\left(x_1^{n_1},x_2^{n_2},...,x_d^{n_d};A\right)$ doesn't depend on n_d. So we have

$$I\left(x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}; A\right) = I\left(x_1^{n_1}, x_2^{n_2}, \ldots, x_d; A\right).$$

ii) For any $(n_2, \ldots, n_d) \in \mathbb{N}^{d-1}$, we have

$$I\left(x_2^{n_2}, \ldots, x_d^{n_d}; 0 :_A x_1\right) = I\left(x_1, x_2^{n_2}, \ldots, x_d^{n_d}; A\right) - e\left(x_2^{n_2}, \ldots, x_d^{n_d}; A/x_1A\right).$$

Because $x_1, x_2, \ldots, x_d$ is an *m*-sequence, by Lemma 2.7 we have $x_2^{n_2}A \subseteq x_2A \subseteq x_1A$. Hence, $x_2^{n_2}\left(A/x_1A\right) = 0$. So that $e\left(x_2^{n_2}, \ldots, x_d^{n_d}; A/x_1A\right) = 0$.

This deduce $I\left(x_2^{n_2}, \ldots, x_d^{n_d}; 0 :_A x_1\right) = I\left(x_1, x_2^{n_2}, \ldots, x_d^{n_d}; A\right)$. □

Next, we give some example for nice *m*-s.o.p.

Remark 3.2.

i) Let A be an co-Cohen-Macaulay R-module. Then every s.o.p of A is a nice m-s.o.p.

ii) Let A be an co-Buchsbaum R-module. Then every s.o.p of A is a nice m-s.o.p.

iii) Let A be an generalized co-Cohen-Macaulay R-module. Then there exists an m-primary ideal q such that every s.o.p contain in q is a nice m-s.o.p.

Proof.

i) As *A* is an co-Cohen-Macaulay module, $I(A) = \sup_{\underline{x}} I(\underline{x}; A) = 0$ with $\underline{x}$ run over all s.o.p of *A*. From Theorem 2.8, we get $\underline{x}$ is a nice *m*-s.o.p.

ii) As *A* is an co-Buchsbaum module, $I(\underline{x}; A)$ is a constant (not depending on s.o.p $\underline{x}$ of *A*). From Theorem 2.8, we get $\underline{x}$ is a nice *m*-s.o.p.

iii) As *A* is an generalized co-Cohen-Macaulay *R*-module, $\ell_R\left(H_i^m(A)\right) < \infty$ for all $i < d$. Thus, from Lemma 2.5 and Theorem 2.8, there exists an *m*-primary ideal *q* such that every s.o.p contain in *q* is a nice *m*-s.o.p. □

Finally, we continue studying the effect of a nice *m*-s.o.p on the annihilation of local homology modules of *A*.

Proposition 3.3. Assume $(Ann_R A)\hat{R} = Ann_{\hat{R}} A$ *and* $\underline{x} = (x_1, x_2, \ldots, x_d)$ *is a s.o.p and a strong m-sequence of Artinian R-module A. Then*

$$x_j H_i^m(A) = 0 \text{ for all } 0 \le i < j \le d.$$

Proof. We proceed by introduction on d = N-dim$A$.

For d = 1 and let x_1 be a s.o.p of A. Because of A is an Artinian R-module, the system $\{m^t A\}$ is stationary, i.e there exists a positive interger s such that $m^t A = m^s A$, for all $t \geq s$.

It follows from Lemma 2.4 that $x_1 H_0^m(A) = x_1 A / m^s A$.

Since x_1 is m-sequence for A and $x_1 A = x_1^s A$, we have $x_1 A \subset m^s A$. This implies $x_1 H_0^m(A) = 0$.

Assume that $d > 1$ and our assertion is true for all Artinian R-module of N-dim smaller than d.

First, we shall prove $x_j H_0^m(A) = 0$ for all $1 \leq j \leq d$. Similar proof in case d = 1, from Lemma 2.7, we get $x_j A \subset x_1 A = x_1^s A \subset m^s A$.

Next, we shall prove $x_j H_i^m(A) = 0$ for all $1 \leq i < j \leq d$.

According to Lemma 2.1 and Lemma 2.2, there exists $k \in \{1, ..., d\}$ such that x_k is a pseudo-A-coregular element and $\ell_R(A/x_k A) < \infty$. Since $x_1, x_2, ..., x_d$ is a s.o.p and a strong m-sequence of A, we have $x_k A \subset x_1 A$. Thus $\ell_R(A/x_1 A) < \ell_R(A/x_k A) < \infty$. This deduces that N-dim$(A/x_1 A) \leq 0$. So $H_i^m(A/x_1 A) = 0$ for all $i > 0$.

The exact sequence $0 \to x_1 A \to A \to A/x_1 A \to 0$ generates the long exact sequence

$$\cdots \to H_{i+1}^m(A/x_1 A) \to H_i^m(x_1 A) \to H_i^m(A) \to H_i^m(A/x_1 A) \to \cdots$$

Since $H_{i+1}^m(A/x_1 A) = H_i^m(A/x_1 A) = 0$ we have $H_i^m(x_1 A) \cong H_i^m(A)$ for all $i > 0$. Moreover, because $x_1, x_2, ..., x_d$ is an m-sequence of A, we get $x_1 A = x_1^n A$ for all $n > 0$. This deduces

$$H_i^m(A) \cong H_i^m(x_1 A) \cong H_i^m(x_1^n A) \text{ for all } i, n > 0.$$

Combining this result and the exact sequence $0 \to 0 :_A x_1^n \to A \to x_1^n A \to 0$ we have the long exact sequence:

$$\cdots \to H_i^m(A) \xrightarrow{x_1^n} H_i^m(A) \xrightarrow{\Delta_i} H_{i-1}^m(0 :_A x_1^n) \to H_{i-1}^m(A) \xrightarrow{x_1^n} H_{i-1}^m(A) \to \cdots$$

Since $Ker\Delta_i = \operatorname{Im} x_1^n = x_1^n H_i^m(A), \forall\, n > 0$ we have $\operatorname{Im}\Delta_i \cong \dfrac{H_i^m(A)}{Ker\Delta_i} = \dfrac{H_i^m(A)}{x_1^n H_i^m(A)}$.

As $x_1, x_2, ..., x_d$ is a strong *m*-sequence of *A* then $x_2, ..., x_d$ is a strong *m*-sequence of $0 :_A x_1^n$. Applying the inductive hypothesis for $0 :_A x_1^n$ to have $x_j H_{i-1}^m\left(0 :_A x_1^n\right) = 0$ for all $1 \le i < j \le d$. Therefore $x_j \operatorname{Im}\Delta_i = 0$. Combining this result and Lemma 2.3 we get

$$x_j H_i^m(A) \subseteq x_1^n H_i^m(A) \subseteq m^n H_i^m(A), \forall n > 0 \Rightarrow x_j H_i^m(A) \subseteq \bigcap_{n>0} m^n H_i^m(A) = 0.$$

Our proof is complete. □

Corollary 3.4. Assume $\left(Ann_R A\right)\hat{R} = Ann_{\hat{R}} A$.

i) Let $x_1, x_2, ..., x_d$ *be a s.o.p and strong* m *-sequence of A. Then*

$$x_j H_i^m\left(0 :_A \left(x_1^{n_1}, ..., x_k^{n_k}\right)\right) = 0 \text{ for all } 0 \le i,\ k < j \le d.$$

ii) Let $x_1, x_2, ..., x_d$ *be a nice m-s.o.p of A. Then*

$$x_j H_i^m\left(0 :_A \left(x_k^{n_k}, ..., x_d^{n_d}\right)\right) = 0 \text{ for all } 0 \le i < j < k \le d.$$

Proof.

i) Because $x_1, x_2, ..., x_d$ is a s.o.p and a strong *m*-sequence of *A*, $x_1^{n_1}, ..., x_k^{n_k}, x_{k+1}^{n_{k+1}}, ..., x_d^{n_d}$ is also a s.o.p and an *m*-sequence of *A* for all $n_1, n_2, ..., n_d \in \mathbb{N}$. Therefore $x_{k+1}, ..., x_d$ is a s.o.p and a strong *m*-sequence of $0 :_A \left(x_1^{n_1}, ..., x_k^{n_k}\right)$.

By Proposition 3.3, we have $x_j H_i^m\left(0 :_A \left(x_1^{n_1}, ..., x_k^{n_k}\right)\right) = 0$ for all $0 \le i,\ k < j \le d$.

ii) Because $x_1, x_2, ..., x_d$ is a nice s.o.p, $x_1, ..., x_{k-1}$ is also a s.o.p and a strong *m*-sequence of $0 :_A \left(x_k^{n_k}, ..., x_d^{n_d}\right)$ for all $n_k, ..., n_d \in \mathbb{N}$.

By Proposition 3.3, we have $x_j H_i^m\left(0 :_A \left(x_k^{n_k}, ..., x_d^{n_d}\right)\right) = 0$. □

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