ĐẠI HỌC THỦ DẦU MỘT
2009 THU DAU MOT UNIVERSITY

Thu Dau Mot University Journal of Science
ISSN 2615 - 9635
journal homepage: ejs.tdmu.edu.vn

High-order iterative scheme for Kirchhoff-type wave equation with the source containing three unknown values

by ***Le Thi Mai Thanh*** *(Department of Basic Science, Nguyen Tat Thanh University),*
Pham Nguyen Nhat Khanh *(Department of Basic Science, Nguyen Tat Thanh University)*
Article Info: *Received Feb. 22nd, 2023, Accepted May 15th, 2023, Available online June 15th, 2023*

Corresponding author: ltmthanh@ntt.edu.vn
https://doi.org/10.37550/tdmu.EJS/2023.05.428

ABSTRACT
In this paper, a high-order iterative scheme is established in order to get a convergent sequence at a rate of order N, ($N \geq 2$) to a local unique weak solution of a nonlinear Kirchhoff-type wave equation associated with Robin conditions.
Keywords: *Faedo-Galerkin method; Robin conditions; High-order iterative scheme; Kirchhoff-type equation.*

1 Introduction

In this paper, we consider the following problem for a Kirchhoff-type wave equation with the source containing three unknown values

$$\begin{cases} u_{tt} - \mu(t, \|u_x(t)\|^2)u_{xx} \\ \qquad = f\left(x, t, u, u(0,t), u(\eta,t), u(1,t)\right), \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions specified later and $h_0 > 0$, $h_1 \geq 0$, $\eta \in [0,1]$ are given constants.

Suppose that problem (1.1) has a unique weak solution u in a Banach space X, defined in the subsection 3.1. The sequence $\{u_m\}$ defined by

$$\begin{aligned} & u_m'' - \mu(t, \|u_{mx}(t)\|^2)u_{mxx} \\ & = \sum_{0 \leq i+j+r+s \leq N-1} \bar{D}^{ijrs} f[u_{m-1}](x,t) \left(u_m(x,t) - u_{m-1}(x,t)\right)^i \left(u_m(0,t) - u_{m-1}(0,t)\right)^j \\ & \qquad \times \left(u_m(\eta,t) - u_{m-1}(\eta,t)\right)^r \left(u_m(1,t) - u_{m-1}(1,t)\right)^s , \end{aligned} \tag{1.2}$$

$0 < x < 1$, $0 < t < T$, *where*

$$f[u_{m-1}](x,t) = f(x, t, u_{m-1}(x,t), u_{m-1}(0,t), u_{m-1}(\eta,t), u_{m-1}(1,t)),$$

$$\bar{D}^{ijrs} f[u_{m-1}](x,t) = \frac{1}{i!j!r!s!} D_3^i D_4^j D_5^r D_6^s f[u_{m-1}](x,t),$$

is said to admit a order N convergence to u if it satisfies the following estimation

$$\|u_m - u\|_X \leq C\,\|u_{m-1} - u\|_X^N. \tag{1.3}$$

The details of the high-order convergent concept can be found in several references, see in [1], [11], [14] and the citations therein.

In the case that the nonlinear term μ of $(1.1)_1$ includes the Kirchhoff term $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)dx$, that is $\mu = \mu\left(t, \|u_x(t)\|^2\right)$, and $f \equiv f(x,t,u(x,t))$, Truong et al. [16] studied a Kirchhoff-Carrier type wave equation, in which a high-order nonlinear iterative sequence $\{u_m\}$ associated with the proposed problem is defined by

$$\begin{aligned} &u_m'' - \mu\left(t, \|u_m(t)\|^2, \|u_{mx}(t)\|^2\right) u_{mxx} \\ &= \sum_{i=0}^{N} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}\left(x,t,u_{m-1}\right)\left(u_m - u_{m-1}\right)^i,\ 0<x<1,\ 0<t<T. \end{aligned} \tag{1.4}$$

By using the Faedo-Galerkin method and the arguments of compactness, the authors proved the existence of weak solution and established a high-order convergence of the sequence $\{u_m\}$ defined by (1.4) to the weak solution of the problem.
When $\mu = 1,$ the problem (1.1) was investigated by Nhan et al. in [10], in which a nonlinear iterative sequence $\{u_m\}$ was defined similarly to (1.2) in the case $f \equiv f(x,t,u(x,t),u(0,t),u(1,t))$. Using the methods employed in [16], the authors established the inequality (1.3) which led to the following error estimation describing the high-order convergent rate of the sequence $\{u_m\}$ to the weak solution of the corresponding problem

$$\|u_m - u\|_X \leq C\gamma^{N^m}, \text{ for all } m \in \mathbb{N}, \tag{1.5}$$

where $C > 0$ and $0 < \gamma < 1$ are constants.

In the equation $(1.1)_1$, the right-hand side nonlinear term contains the unknown values $u(0,t)$, $u(\eta,t)$ and $u(1,t)$, also known as nonlocal terms. Previously, there have been a lot of published studies of the problems with nonlocal terms, however, the results mainly focused on ordinary differential equations, only few of them are on partial differential equations. One result can be mentioned here such as the work of Pellicer and Morales [12], in which they considered the strongly damped one-dimensional wave equation with dynamic boundary conditions

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1,t), \dfrac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon\left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right] - \varepsilon f\left(u(1,t), \dfrac{u_t(1,t)}{\sqrt{\varepsilon}}\right), \end{cases} \tag{1.6}$$

with $x \in (0,1)$, $t > 0$, α, $r > 0$ and $\varepsilon \geq 0$. The problem (1.6) describes a mass-spring damper model, where the term $\varepsilon f\left(u(1,t), \dfrac{u_t(1,t)}{\sqrt{\varepsilon}}\right)$ represents the control acceleration at $x = 1$. Using the multi-scale invariant theory, the authors proved that the solution of (1.6) for small values of the parameter ε converges to a two-dimensional invariant manifold. It is also necessary to mention the results in [2] and [9], in which the authors investigated the unique solvability of nonlinear wave equations with nonlocal terms $u(0,t)$, $u(1,t)$ and $u(\eta_1,t),\cdots,u(\eta_q,t)$ respectively, where $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq 1$. Recently, several published results of high-order iterative schemes for some equations with integral terms have been interested, see in [6] with the integral term $\|u(t)\|^2 = \int_0^1 u^2(x,t)dx$, in [7] with the integral term $\int_0^t g(x,t,s,u(x,s))ds$ and in [13] with the integral term $\int_0^t g(t-s)\frac{\partial^2}{\partial x^2}\left(\bar{\mu}(x,s,u(x,s))\right)ds$. In the papers, the authors have also constructed the nolinear iterative schemes corresponding to the considered problems and proved the high-order convergences of the schemes to the solutions.

Based on the ideas of the high-order iterative method used in the above papers, in this paper, we prove the existence of the high-order iterative scheme given by (1.2). Note that the high-order iterative scheme (1.2) connects to the problem (1.1) by using Taylor expansion of the multi-variable function

$$f = f(x,t,u,u(0,t),u(\eta,t),u(1,t));$$

the such construction of high-order iterative schemes has not been mentioned in the previous papers. Furthermore, an error estimation on the convergent rate of the scheme has also been established as in (1.5). The main results of our paper are presented in Theorem 3.1 and Theorem 4.1. The proofs of these results rely on the fixed point method, the Faedo-Galerkin method, and the arguments related to compactness. This paper can be considered as a relative generalization of the results in [2], [4], [6], [8], [10] and [16].

2 Preliminaries

First, we put $\Omega = (0,1)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot,\cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, and

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T ||u(t)||_X^p dt\right)^{1/p}, & \text{for} \quad 1 \le p < \infty, \\ \underset{0<t<T}{ess\sup} ||u(t)||_X, & \text{for} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R}^4)$, $f = f(x,t,y_1,\cdots,y_4)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{2+i} f = \frac{\partial f}{\partial y_i}$, $i = 1,2,3,4$ and $D^\alpha f = D_1^{\alpha_1}\cdots D_6^{\alpha_6} f$; $\alpha = (\alpha_1,\cdots,\alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 = k$, $D^{(0,\cdots,0)} f = D^0 f = f$.

On H^1, we use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

We put

$$a(u,v) = \int_0^1 u_x(x)v_x(x)dx + h_0 u(0)v(0) + h_1 u(1)v(1),\ u,v \in H^1. \tag{2.1}$$

We have the following lemmas, of which the proofs are straightforward, so we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact, and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\,\|v\|_{H^1} \text{ for all } v \in H^1. \tag{2.2}$$

Lemma 2.2. *Let $h_0,\, h_1 \geq 0$, with $h_0 + h_1 > 0$. Then, the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.1) is continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*

$$\begin{aligned} &(i) \quad |a(u,v)| \leq a_1 \|u\|_{H^1} \|v\|_{H^1}, \\ &(ii) \quad a(v,v) \geq a_0 \|v\|_{H^1}^2, \end{aligned} \tag{2.3}$$

for all $u,\, v \in H^1$, where $a_1 = 1 + 2h_0 + 2h_1$ and

$$a_0 = \frac{1}{4}\min\{1,\ \max\{h_0,\ h_1\}\}. \tag{2.4}$$

Remark 2.3. *From (2.3), it follows that on H^1, $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are two equivalent norms satisfying*

$$\sqrt{a_0}\,\|v\|_{H^1} \leq \|v\|_a \leq \sqrt{a_1}\,\|v\|_{H^1},\ \forall v \in H^1. \tag{2.5}$$

3 Main results

3.1 The high-order iterative schemes

We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$;

(H_2) $\mu \in C^1(\mathbb{R}_+^2;\mathbb{R})$ there exists a constant $\mu_* > 0$ such that
$$\mu(t,z) \geq \mu_*,\ \forall (t,z) \in \mathbb{R}_+^2;$$

(H_3) $f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$ such that

(i) $D_3^i D_4^j D_5^r D_6^s f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $0 \leq i+j+r+s \leq N$,

(ii) $D_1 D_3^i D_4^j D_5^r D_6^s f,\ D_3^{i+1} D_4^j D_5^r D_6^s f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$,
$$1 \leq i+j+r+s \leq N-1.$$

Fix a constant $T^* > 0$, for each $T \in (0, T^*]$, we put

$$\begin{aligned} W_T &= \{v \in L^\infty(0,T;H^2) : v' \in L^\infty(0,T;H^1),\ v'' \in L^2(Q_T)\}, \\ W_1(T) &= C([0,T];H^1) \cap C^1([0,T];L^2), \end{aligned} \tag{3.1}$$

as two Banach spaces with respect to the norms (see Lions [3])

$$\begin{aligned} \|v\|_{W_T} &= \max\{\|v\|_{L^\infty(0,T;H^2)}, \|v'\|_{L^\infty(0,T;H^1)}, \|v''\|_{L^2(Q_T)}\}, \\ \|v\|_{W_1(T)} &= \|v\|_{C([0,T];H^1)} + \|v'\|_{C^0([0,T];L^2)}. \end{aligned} \tag{3.2}$$

We define that a function $u = u(x,t)$ is a weak solution of the problem (1.1) if $u \in \tilde{W}_T = \{v \in L^\infty(0,T;H^2) : v' \in L^\infty(0,T;H^1),\ v'' \in L^\infty(0,T;L^2)\}$, and u satisfies the following variational equation

$$\begin{cases} \langle u''(t), w\rangle + \mu[u](t)a(u(t),w) = \langle f[u](t), w\rangle,\ \forall w \in H^1, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \end{cases} \tag{3.3}$$

where

$$\mu[u](t) = \mu(t, \|u_x(t)\|^2),$$
$$f[u](x,t) = f\left(x, t, u, u(0,t), u(\eta,t), u(1,t)\right). \tag{3.4}$$

For every $M > 0$, we denote

$$\begin{cases} W(M,T) = \{v \in W_T : \|v\|_{W_T} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}. \end{cases} \tag{3.5}$$

Now, we establish the recurrent sequence $\{u_m\}$.
The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W(M,T). \tag{3.6}$$

We find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the nonlinear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \bar{\mu}_m(t) a(u_m(t), w) = \langle F_m(t), w\rangle, \ \forall w \in H^1, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.7}$$

where

$$\bar{\mu}_m(t) = \mu[u_m](t) = \mu(t, \|u_{mx}(t)\|^2), \tag{3.8}$$
$$F_m(x,t) = \tag{3.9}$$
$$\sum_{i+j+r+s\leq N-1} \bar{D}^{ijrs} f[u_{m-1}](x,t) \left(u_m(x,t) - u_{m-1}(x,t)\right)^i \left(u_m(0,t) - u_{m-1}(0,t)\right)^j$$
$$\times \left(u_m(\eta,t) - u_{m-1}(\eta,t)\right)^r \times \left(u_m(1,t) - u_{m-1}(1,t)\right)^s,$$

with the notations

$$\bar{D}^{ijrs} f[u](x,t) = \frac{1}{i!j!r!s!} D_3^i D_4^j D_5^r D_6^s f[u](x,t), \ 1 \leq i+j+r+s \leq N, \tag{3.10}$$
$$\bar{D}^{0000} f[u](x,t) = f[u](x,t).$$

Then we have the following theorem.

Theorem 3.1. *Let $(H_1) - (H_3)$ hold. Then there exist constants M, T such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by (3.6) - (3.8).*

The proof of Theorem 3.1 is based on the Faedo-Galerkin method introduced by Lions [3] and on the similar techniques given in [2], [10] and [16].

3.2 Convergence of scheme and error estimation

In this section, we prove the convergence of the scheme (1.2). Moreover, we also establish the error estimation describing the N-order convergent rate of the scheme (1.2) to the weak solution u of the problem (1.1). The desired results are presented in the following theorem.

Theorem 3.2. *Let $(H_1)-(H_3)$ hold. Then, there are two constants $M>0$ and $T>0$ as in Theorem 3.1, such that the problem (1.1) admits a unique weak solution $u \in W_1(M,T)$ and the recurrent sequence $\{u_m\}$ defined by (3.6) - (3.8) converges strongly with a N-order rate to the solution u in the space $W_1(T)$ and in the sense*

$$\|u_m - u\|_{W_1(T)} \leq C\,\|u_{m-1}-u\|^N_{W_1(T)}, \tag{3.11}$$

for all $m \geq 1$, where C is a suitable constant. On the other hand, the following estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \leq C_T\,(k_T)^{N^m}, \text{ for all } m \in \mathbb{N}, \tag{3.12}$$

where C_T and $0<k_T<1$ are constants depending only on T.

Proof. (i) *Existence of the solution.* We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Indeed, let $v_m = u_{m+1}-u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + \bar{\mu}_{m+1}(t)a(v_m(t),w) \\ \qquad = (\bar{\mu}_{m+1}(t)-\bar{\mu}_m(t))\,\langle u_{mxx}(t),w\rangle + \langle F_{m+1}(t)-F_m(t),w\rangle\,,\ \forall w \in H^1, \\ v_m(0)=v_m'(0)=0. \end{cases} \tag{3.13}$$

Taking $w=v_m'$ in (3.13), after integrating in t, we get

$$\begin{aligned} \gamma_*\rho_m^*(t) \leq & \int_0^t \bar{\mu}_{m+1}'(s)\,\|v_m(s)\|_a^2\,ds + 2\int_0^t (\bar{\mu}_{m+1}(s)-\bar{\mu}_m(s))\,\langle u_{mxx}(s), v_m'(s)\rangle ds \\ & + 2\int_0^t \langle F_{m+1}(s)-F_m(s), v_m'(s)\rangle\,ds \\ = & \ z_1+z_2+z_3, \end{aligned} \tag{3.14}$$

where $\gamma_* = \min\{1, \mu_* a_0\}$ and

$$\rho_m^*(t) = \|v_m'(t)\|^2 + \|v_m(t)\|^2_{H^1}. \tag{3.15}$$

Next, we shall estimate the integrals on the right-hand side of (3.14) as below.
We denote two constants $\tilde{K}_M(\mu)$, $K_M(f)$ by

$$\begin{aligned} \tilde{K}_M(\mu) &= \|\mu\|_{C^1([0,T^*]\times[0,M^2])} = \|\mu\|_{C([0,T^*]\times[0,M^2])} \\ &\quad + \|D_1\mu\|_{C([0,T^*]\times[0,M^2])} + \|D_2\mu\|_{C([0,T^*]\times[0,M^2])}, \\ K_M(f) &= \sum_{i+j+r+s\leq N} \left\|D_3^iD_4^jD_5^rD_6^sf\right\|_{C(D_M)} \\ &\quad + \sum_{1\leq i+j+r+s\leq N-1} \left\|D_1D_3^iD_4^jD_5^rD_6^sf\right\|_{C(D_M)} \\ &\quad + \sum_{1\leq i+j+r+s\leq N-1} \left\|D_3^{i+1}D_4^jD_5^rD_6^sf\right\|_{C(D_M)}, \\ \|\mu\|_{C([0,T^*]\times[0,M^2])} &= \sup_{(t,z)\in[0,T^*]\times[0,M^2]} |\mu(t,z)|, \\ \|f\|_{C(D_M)} &= \sup_{(x,t,y_1,y_2,y_3,y_4)\in D_M} |f(x,t,y_1,y_2,y_3,y_4)|, \\ D_M &= \{(x,t,y_1,y_2,y_3,y_4)\in[0,1]\times[0,T^*]\times\mathbb{R}^4 : \max_{1\leq i\leq 4}|y_i|\leq M\}. \end{aligned} \tag{3.16}$$

Estimation of z_1. We have

$$\bar{\mu}'_{m+1}(t) = D_1\mu(t, \|\nabla u_{m+1}(t)\|^2) + 2D_2\mu(t, \|\nabla u_{m+1}(t)\|^2)\langle \nabla u_{m+1}(t), \nabla u'_{m+1}(t)\rangle,$$

Hence

$$\left|\bar{\mu}'_{m+1}(t)\right| \leq \tilde{K}_M(\mu)(1 + 2M^2). \tag{3.17}$$

So z_1 is estimated as follows

$$\begin{aligned} z_1 &= \int_0^t \bar{\mu}'_{m+1}(s)\, \|v_m(s)\|_a^2\, ds \leq \tilde{K}_M(\mu)(1 + 2M^2)a_1 \int_0^t \|v_m(s)\|_{H^1}^2\, ds \\ &\leq \tilde{K}_M(\mu)(1 + 2M^2)a_1 \int_0^t \rho_m^*(s)ds. \end{aligned} \tag{3.18}$$

Estimation of z_2. Due to (H_2) and (3.14), we have

$$\begin{aligned} |\bar{\mu}_{m+1}(s) - \bar{\mu}_m(s)| &= \left|\mu(s, \|\nabla u_{m+1}(s)\|^2) - \mu(t, \|\nabla u_m(s)\|^2)\right| \\ &\leq \tilde{K}_M(\mu)\left|\|\nabla u_{m+1}(s)\|^2 - \|\nabla u_m(s)\|^2\right| \\ &\leq 2M\tilde{K}_M(\mu)\, \|\nabla v_m(s)\| \leq 2M\tilde{K}_M(\mu)\, \|v_m(s)\|_a \\ &\leq 2M\tilde{K}_M(\mu)\sqrt{a_1}\, \|v_m(s)\|_{H^1}. \end{aligned} \tag{3.19}$$

Then, z_2 is estimated as follows

$$\begin{aligned} z_2 &= 2\int_0^t (\bar{\mu}_{m+1}(s) - \bar{\mu}_m(s))\, \langle u_{mxx}(s), v'_m(s)\rangle ds \\ &\leq 2\int_0^t |\bar{\mu}_{m+1}(s) - \bar{\mu}_m(s)|\, \|u_{mxx}(s)\|\, \|v'_m(s)\|\, ds \\ &\leq 4M^2\tilde{K}_M(\mu)a_1 \int_0^t \|v_m(s)\|_{H^1}\, \|v'_m(s)\|\, ds \\ &\leq 2M^2\tilde{K}_M(\mu)a_1 \int_0^t \rho_m^*(s)ds. \end{aligned} \tag{3.20}$$

Estimation of z_3. By using Taylor's expansion of the function

$$f(x, t, u_m(x, t), u_m(0, t), u_m(\eta, t), u_m(1, t)) = f\,[u_m] = f\,[u_{m-1} + v_{m-1}]$$

around the point $(x, t, u_{m-1}, u_{m-1}(x, t), u_{m-1}(0, t), u_{m-1}(\eta, t), u_{m-1}(1, t))$ up to N^{th} order, we obtain

$$\begin{aligned} &f\,[u_m]\,(x, t) - f\,[u_{m-1}]\,(x, t) \\ &= f(x, t, u_{m-1}, u_m(x, t), u_m(0, t), u_m(\eta, t), u_m(1, t)) \\ &\quad - f(x, t, u_{m-1}(x, t), u_{m-1}(0, t), u_{m-1}(\eta, t), u_{m-1}(1, t)) \\ &= \sum_{1\leq i+j+r+s\leq N-1} D^{ijrs} f[u_{m-1}](x, t)v_{m-1}^i(x, t)v_{m-1}^j(0, t)v_{m-1}^r(\eta, t)v_{m-1}^s(1, t) \\ &\quad + \sum_{i+j+r+s=N} D^{ijrs} f[u_{m-1} + \theta v_{m-1}](x, t)v_{m-1}^i(x, t)v_{m-1}^j(0, t)v_{m-1}^r(\eta, t)v_{m-1}^s(1, t), \end{aligned} \tag{3.21}$$

where $0 < \theta < 1$.

On the other hand, due to

$$\begin{aligned} & F_{m+1}(x,t)-F_m(x,t) \\ & = f[u_m]-f[u_{m-1}] \\ & \quad - \sum_{1\le i+j+r+s\le N-1} D^{ijrs}f[u_{m-1}](x,t)v_{m-1}^i(x,t)v_{m-1}^j(0,t)v_{m-1}^r(\eta,t)v_{m-1}^s(1,t) \\ & \quad + \sum_{1\le i+j+r+s\le N-1} D^{ijrs}f[u_m](x,t)v_m^i(x,t)v_m^j(0,t)v_m^r(\eta,t)v_m^s(1,t) \end{aligned} \tag{3.22}$$

then we deduce from (3.8) and (3.22) that

$$\begin{aligned} & F_{m+1}(x,t)-F_m(x,t) \\ & = \sum_{1\le i+j+r+s\le N-1} D^{ijrs}f[u_m](x,t)v_m^i(x,t)v_m^j(0,t)v_m^r(\eta,t)v_m^s(1,t) \\ & \quad + \sum_{i+j+r+s=N} D^{ijrs}f[u_{m-1}+\theta v_{m-1}](x,t)v_{m-1}^i(x,t)v_{m-1}^j(0,t)v_{m-1}^r(\eta,t)v_{m-1}^s(1,t). \end{aligned} \tag{3.23}$$

Using the equality $\sum_{i+j+r+s=p}\frac{1}{i!j!r!s!}=\frac{4^p}{p!}$, $\forall p\in\mathbb{Z}_+$, we deduce from (3.23) that

$$\begin{aligned} & \|F_{m+1}(t)-F_m(t)\| \\ & \le K_M(f)\sum_{1\le i+j+r+s\le N-1}\frac{1}{i!j!r!s!}(\sqrt{2})^{i+j+r+s}\left\|v_m(t)\right\|_{H^1}^{i+j+r+s} \\ & \quad + K_M(f)\sum_{i+j+r+s=N}\frac{1}{i!j!r!s!}(\sqrt{2})^{i+j+r+s}\left\|v_{m-1}(t)\right\|_{H^1}^{i+j+r+s} \\ & = K_M(f)\sum_{p=1}^{N-1}\sum_{i+j+r+s=p}\frac{1}{i!j!r!s!}(\sqrt{2})^{p}\left\|v_m(t)\right\|_{H^1}^{p} \\ & \quad + K_M(f)\sum_{i+j+r+s=N}\frac{1}{i!j!r!s!}(\sqrt{2})^{N}\left\|v_{m-1}(t)\right\|_{H^1}^{N} \\ & = K_M(f)\sum_{p=1}^{N-1}\frac{(4\sqrt{2})^p}{p!}\left\|v_m(t)\right\|_{H^1}^{p} + K_M(f)\frac{(4\sqrt{2})^N}{N!}\left\|v_{m-1}(t)\right\|_{H^1}^{N} \\ & \le \frac{1}{\sqrt{a_0}}K_M(f)\sum_{p=1}^{N-1}\frac{(4\sqrt{2})^p\,(2M)^{p-1}}{p!}\sqrt{\rho_m^*(t)} + K_M(f)\frac{(4\sqrt{2})^N}{N!}\left\|v_{m-1}\right\|_{W_1(T)}^{N} \\ & \equiv \gamma_T^{(1)}\sqrt{\rho_m^*(t)}+\gamma_T^{(2)}\left\|v_{m-1}\right\|_{W_1(T)}^{N}, \end{aligned} \tag{3.24}$$

where $\gamma_T^{(1)}=\frac{1}{\sqrt{a_0}}K_M(f)\sum_{p=1}^{N-1}\frac{(4\sqrt{2})^p\,(2M)^{p-1}}{p!}$, $\gamma_T^{(2)}=\frac{(4\sqrt{2})^N}{N!}K_M(f)$.

Then we deduce, from (3.24) that

$$
\begin{aligned}
z_3 &= 2\int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s)\rangle\, ds \leq 2\int_0^t \|F_{m+1}(s) - F_m(s)\| \, \|v_m'(s)\|\, ds \qquad (3.25)\\
&\leq 2\int_0^t \left(\gamma_T^{(1)}\sqrt{\rho_m^*(s)} + \gamma_T^{(2)}\|v_{m-1}\|_{W_1(T)}^N\right)\sqrt{\rho_m^*(s)} ds\\
&\leq 2\int_0^t \left(\gamma_T^{(1)}\rho_m^*(s) + \gamma_T^{(2)}\|v_{m-1}\|_{W_1(T)}^N\sqrt{\rho_m^*(s)}\right) ds\\
&= 2\int_0^t \gamma_T^{(1)}\rho_m^*(s)ds + \gamma_T^{(2)}\left[T\|v_{m-1}\|_{W_1(T)}^{2N} + \int_0^t \rho_m^*(s)ds\right]\\
&= T\gamma_T^{(2)}\|v_{m-1}\|_{W_1(T)}^{2N} + \gamma_T^{(3)}\int_0^t \rho_m^*(s)ds,
\end{aligned}
$$

where $\gamma_T^{(3)} = 2\gamma_T^{(1)} + \gamma_T^{(2)}$.

Combining z_1, z_2, z_3 and (3.14), it leads to

$$\rho_m^*(t) \leq T\theta_M^{(1)}\|v_{m-1}\|_{W_1(T)}^{2N} + 2\theta_M^{(2)}\int_0^t \rho_m^*(s)ds, \qquad (3.26)$$

where

$$\theta_M^{(1)} = \frac{1}{\gamma_*}\gamma_T^{(2)}, \quad \theta_M^{(2)} = \frac{1}{2\gamma_*}\left(\gamma_T^{(3)} + a_1\tilde{K}_M(\mu)(1+4M^2)\right).$$

Using Gronwall's lemma, we obtain from (3.26) that

$$\|v_m\|_{W_1(T)} \leq \mu_T\|v_{m-1}\|_{W_1(T)}^N, \qquad (3.27)$$

where $\mu_T = 2\sqrt{T\theta_M^{(1)}}\exp\left(T\theta_M^{(2)}\right)$.

Choosing $T > 0$ small enough such that $k_T = M\mu_T^{\frac{-1}{N-1}} < 1$, then it deduces from (3.27) that

$$\|u_m - u_{m+p}\|_{W_1(T)} \leq (1-k_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}(k_T)^{N^m}, \text{ for all } m,\ p \in \mathbb{N}. \qquad (3.28)$$

This follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \qquad (3.29)$$

Notice that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$
\begin{cases}
u_{m_j} \to u & \text{in } L^\infty(0,T;H^2) \text{ weak*},\\
u'_{m_j} \to u' & \text{in } L^\infty(0,T;H^1) \text{ weak*},\\
u''_{m_j} \to u'' & \text{in } L^2(Q_T) \text{ weak},\\
u \in W(M,T).
\end{cases} \qquad (3.30)
$$

We shall prove

$$\mu[u_m](t) \to \mu(t, \|u_x(t)\|^2) \text{ strongly in } C([0,T]), \qquad (3.31)$$

and

$$F_m(t) \to f(\cdot, t, u(t), u(0,t), u(\eta,t), u(1,t)) \text{ strongly in } C([0,T];L^2). \qquad (3.32)$$

The proof of (3.31). Due to (H_2) and (3.14), we have

$$\begin{aligned} \left|\mu[u_m](t) - \mu(t, \|u_x(t)\|^2)\right| &\leq \tilde{K}_M(\mu) \left|\|\nabla u_m(t)\|^2 - \|u_x(t)\|^2\right| \\ &\leq 2M\tilde{K}_M(\mu) \|\nabla u_m(t) - u_x(t)\| \\ &\leq 2M\tilde{K}_M(\mu) \|u_m - u\|_{W_1(T)} . \end{aligned} \tag{3.33}$$

Hence, we deduce from (3.29) and (3.33) that (3.31) is hold.
The proof of (3.32). Due to (H_3) and (3.14), we have

$$\begin{aligned} &\|F_m(\cdot,t) - f(\cdot,t,u(t),u(0,t),u(\eta,t),u(1,t))\| \leq \\ &\|f(\cdot,t,u_{m-1}(t),u_{m-1}(0,t),u_{m-1}(\eta,t),u_{m-1}(1,t)) - f(\cdot,t,u(t),u(0,t),u(\eta,t),u(1,t))\| \\ &\quad + \sum_{1\leq i+j+r+s\leq N-1} \Big\| D^{ijrs} f[u_{m-1}] \left(u_m(t) - u_{m-1}\right)^i \left(u_m(0,t) - u_{m-1}(0,t)\right)^j \\ &\qquad\qquad \times \left(u_m(\eta,t) - u_{m-1}(\eta,t)\right)^r \left(u_m(1,t) - u_{m-1}(1,t)\right)^s \| \\ &\leq 4K_M(f) \|u_{m-1} - u\|_{W_1(T)} + \\ &K_M(f) \sum_{1\leq i+j+r+s\leq N-1} \frac{1}{i!j!r!s!} (\sqrt{2})^{i+j+r+s} \|u_m - u_{m-1}\|_{W_1(T)}^{i+j+r+s} \\ &\leq 4K_M(f) \|u_{m-1} - u\|_{W_1(T)} + K_M(f) \sum_{p=1}^{N-1} \frac{(4\sqrt{2})^p}{p!} \|u_m - u_{m-1}\|_{W_1(T)}^p . \end{aligned} \tag{3.34}$$

Thus, we deduce from (3.29) and (3.34) that (3.32) is valid.
Finally, taking the limitations in (3.7) as $m = m_j \to \infty$ and using (3.31) and (3.32), we conclude that there exists $u \in W(M,T)$ satisfying the variational equation

$$\langle u''(t), w\rangle + \mu(t, \|u_x(t)\|^2) a(u(t), w) = \langle f(\cdot,t,u(t),u(0,t),u(\eta,t),u(1,t)), w\rangle , \tag{3.35}$$

for all $w \in H^1$ and initial conditions

$$u(0) = \tilde{u}_0, \;\; u'(0) = \tilde{u}_1. \tag{3.36}$$

On the other hand, due to $(3.33)_4$ and (3.35), we get that

$$u'' = \Delta u + f(x,t,u(t),u(0,t),u(1,t)) \in L^\infty(0,T;L^2), \tag{3.37}$$

hence $u \in W_1(M,T)$.
Uniqueness. Apply the same arguments used in the proof of Theorem 3.1, the uniqueness is confirmed. Therefore, $u \in W_1(M,T)$ is the unique weak solution of the problem (1.1).
Note that the estimation (3.11) is easy to be deduced from the estimation (3.27). Moreover, taking the limitation in (3.28) as $p \to \infty$ (with a fixed m), we get (3.12). Theorem 3.2 is completely proven. □

Remark 3.3. *To construct a N-order iterative algorithm, we need to assume (H_3). Then we obtain the existence and the N-order convergence of the scheme (1.2). The assumption (H_3) of the function f can be relaxed if we consider the existence of solution only, see [4] and [9].*

Acknowledgment. The authors wish to express their sincere thanks to the editor and the referees for the valuable comments and important remarks for the improvement of the paper.

References

[1] K. Deimling (1985). Nonlinear Functional Analysis, *Springer, New York.*

[2] P.N.N. Khanh, L.T.M. Thanh, T.T.M. Dung, N.H. Nhan (2021). *Numerical results via high-order iterative scheme to a nonlinear wave equation with source containing two unknown boundary values*, In: P. Cong Vinh, N. Huu Nhan (eds). Nature of Computation and Communication. ICTCC 2021. Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering, vol 408. Springer, Cham. https://doi.org/10.1007/978-3-030-92942-8_17

[3] J.L. Lions (1969). Quelques méthodes de résolution des problèmes aux limites nonlinéaires, *Dunod; Gauthier –Villars, Paris.*

[4] N.T. Long, T.N. Diem (1997). *On the nonlinear wave equation* $u_{tt} - u_{xx} = f(x, t, u, u_x, u_t)$ *associated with the mixed homogeneous conditions*, Nonlinear Anal. TMA. *29*, 1217-1230.

[5] L.T.P. Ngoc, N.T.T. Truc, T.T.H. Nga, N.T. Long (2015). *On a high order iterative scheme for a nonlinear wave equation*, Nonlinear Funct. Anal. Appl. *20*, 123-140.

[6] L.T.P. Ngoc, B.M. Tri, N.T. Long (2017). *An N-order iterative scheme for a nonlinear wave equation containing a nonlocal term*, Filomat, *31*, 1755-1767.

[7] N.H. Nhan, L.T.P. Ngoc, T.M. Thuyet, N.T. Long (2016). *On a high order iterative scheme for a nonlinear wave equation with the source term containing a nonlinear integral*, Nonlinear Funct. Anal. Appl. *21*, 65-84.

[8] N.H. Nhan, N.T. Than, L.T.P. Ngoc, N.T. Long (2017). *Linear approximation and asymptotic expansion associated to the Robin-Dirichlet problem for a nonlinear wave equation with the source term containing an unknown boundary value*, Nonlinear Funct. Anal. Appl. *22*, 403-420.

[9] N.H. Nhan, L.T.P. Ngoc, N.T. Long (2017). *Existence and asymptotic expansion of the weak solution for a wave equation with nonlinear source containing nonlocal term*, Boundary Value Problems,*2017*, Article: 87.

[10] N.H. Nhan, N.T. Than, L.T.P. Ngoc, N.T. Long (2017). *A N-order iterative scheme for the Robin problem for a nonlinear wave equation with the source term containing the unknown boundary values*, Nonlinear Funct. Anal. Appl. *22*, 573-594.

[11] P.K. Parida, D.K. Gupta (2007). *Recurrence relations for a Newton-like method in Banach spaces*, J. Comput. Appl. Math. *206*, 873-887.

[12] M. Pellicer, J. Solà-Morales (2008). *Spectral analysis and limit behaviours in a spring-mass system*, Comm. Pure. Appl. Math. *7*, 563-577.

[13] D.T.N. Quynh, N.H. Nhan, L.T.P. Ngoc, N.T. Long (2022). *On a high-order iteration technique for a wave equation with nonlinear viscoelastic term*, Filomat, *36*, 5765-5794.

[14] Eberhard Zeidler (1985). Nonlinear Functional Analysis and its Applications, *Springer-Verlag NewYork Berlin Heidelberg Tokyo*, Part I.

[15] R.E. Showalter (1994). Hilbert space methods for partial differential equations, *Electron. J. Differ. Equ. Monograph 01*.

[16] L.X. Truong, L.T.P. Ngoc, N.T. Long (2009). *The N - order iterative schemes for a nonlinear Kirchhoff-Carrier wave equation associated with the mixed inhomogeneous conditions*, Appl. Math. Comput. *215*, 1908-1925.