Thu Dau Mot University
Journal of Science
ISSN 2615 - 9635
journal homepage: ejs.tdmu.edu.vn

Identifying the source for Fractional diffusion equations with Random Gaussian white noise

by ***Ngo Ngoc Hung*** *(Faculty of Fundamental Science, Industrial university of Ho Chi Minh City, Ho Chi Minh City, Vietnam),* ***Tran Thanh Binh*** *(Faculty of Mathematics and Applications, Sai Gon University, Ho Chi Minh City, Vietnam),* ***Nguyen Duc Phuong*** *(Faculty of Fundamental Science, Industrial university of Ho Chi Minh City, Ho Chi Minh City)*

Article Info: *Received Feb. 22nd, 2023, Accepted May 15th, 2023, Available online June 15th, 2023*

Corresponding author: nguyenducphuong@iuh.edu.vn
https://doi.org/10.37550/tdmu.EJS/2023.05.426

ABSTRACT

In this work, we study the problem of finding the source function of the inhomogeneous diffusion equations with conformable derivative ${}^c\partial_t^\alpha u - \Delta u = f(x)$, $0 < \alpha < 1$, *associate with random noisy input data. This problem is ill-posed in the sense of Hadamard. In order to regulate the instablity of the solution, we applied the truncation method and estimated the error estimate between the exact solution and the regularized solution.*

Keywords: *Diffusion equations, Regularization, Random noise, Finding source, Conformable derivative*

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$, with sufficiently smooth boundary $\partial\Omega$. In this article, for the equation

$$ {}^c\partial_t^\alpha u(x,t) - \Delta u(x,t) = f(x), \quad x \in \Omega, t \in (0,T) \tag{1.1}$$

accompanied with boundary condition, the initial condition and the terminal condition

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = \rho(x), & x \in \Omega, \\ u(x,T) = \xi(x), & x \in \Omega. \end{cases}$$

Our goal here is recovering the source $f(x)$. The derivative with respect to the time variable ${}^c\partial_t^\alpha$ is in the sense of Conformable derivative with order $\alpha \in (0,1)$ (see [1]).

Definition 1.1 (Conformable derivative)**.** *Given a function* $u : [0,\infty) \to \mathbb{R}$, *the Conformable fractional of order* $\alpha \in (0,1]$ *is defined by*

$$ {}^c\partial_t^\alpha u(t) = \lim_{\epsilon \to 0} \frac{u(t+\epsilon t^{1-\alpha}) - u(t)}{\epsilon}, \tag{1.2}$$

for all $t > 0$. *If* f *is* α *differentiable in* $(0, a), a > 0$, *and the* $\lim_{t \to t_0^+} {}^c\partial_t^\alpha u(t)$ *exists, then*

$$ {}^c\partial_t^\alpha u(t_0) = \lim_{t \to t_0^+} {}^c\partial_t^\alpha u(t). $$

Proposition 1.2 ([1]). *If a function* $u : (a, \infty) \to \mathbb{R}$ *is differentiable at a point* $t > 0$, *then*

$$ \partial_t^\alpha u = t^{1-\alpha} \frac{\mathrm{d}u}{\mathrm{d}t}, \quad \alpha \in (0, 1). \tag{1.3} $$

Fractional diffusion equations there are many applications: In Electrochemistry, they are used to model the diffusion of ions in electrolytes. The fractional derivative accounts for the non-local interactions between the ions, which are not captured by the classical diffusion equation; In Image processing they are used to enhance images by removing noise and preserving edges. The fractional derivative acts as a low-pass filter that smooths the image while preserving the edge; In Finance, they used to model the dynamics of financial assets such as stocks, commodities, and currencies. In Materials science, they are used to captures the memory effect that is observed in financial data.

In practice, it is hard to obtain the precise final value function data and instead we only have its observed values. It is a fact that observations are allway contain random errors, which stem from the limitations of the measuring device (measurement error). Hence, it is natural that observations are often accompanied by some degree of noise. In this paper, we will explore the case where such perturbations take the form of an additive stochastic white noise.

$$ \xi^\epsilon(x) = \xi(x) + \epsilon W(x), \tag{1.4} $$

where ϵ is the amplitude of the noise and $W(x)$ is a Gaussian white noise process. Suppose further that even the observations (1.4) cannot be observed exactly, but they can only be observed in discretized form

$$ \langle \xi^\epsilon, \varphi_j \rangle = \langle \xi, \varphi_j \rangle + \epsilon \langle W, \varphi_j \rangle, \quad j = 1, \dots, n, \tag{1.5} $$

where $\{\varphi_j\}$ is a orthonormal basic of Hilbert space $L^2(\Omega)$; $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$; $W_j := \langle W, \varphi_j \rangle$ follows the standard normal distribution; and $\langle \xi^\epsilon, \varphi_j \rangle$ are independent random variables for orthonormal functions φ_j. For more detail on the white noise model see, [2–4].

Numerous research studies have been conducted on the inverse source problem of a time-fractional diffusion equation. Over the past few decades, mathematicians across the globe have made significant technical advancements in this area, such as the Quasi-Reversibility method (see [5]), Quasi-Boundary Value method (see [6]), the Landweber iterative method (see [7]), the Fractional Landweber method (see [8]), a Tikhonov regularization method [9], and a Fourier truncation method (see [10]).

This Problem with random noise is ill-posed in the sense of Hadamard, therefore regularization methods for it are required. The aim of this paper is to find the approximation for the source ξ from indirect and noisy discrete observations (1.5) and then we use them to propose a regularized solution by the Fourier truncation method.

The organizational structure of this paper is as follows. We first introduce some preliminary materials in Section 2. In Section 3, we give an example of Ill-Posed. In Section 4, we draw into main results: first we propose a new regularized solution, and then we give the convergent estimates between a mild solution and a regularized solution under some priori assumptions on the exact solution. To end this section, we discuses a regularization parameter choice rules.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ be an open bounded domain and let $\langle \cdot, \cdot \rangle$ be the inner product of $L^2(\Omega)$. Then, there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ ($\varphi_j \in H_0^1(\Omega) \cap C^{\infty}(\Omega)$) of $L^2(\Omega)$ consisting of eigenfunctions $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lim_{j\to\infty} \lambda_j = +\infty$ of the Laplacian operator $-\Delta$ in Ω such that $-\Delta\varphi_j(x) = \lambda_j \varphi_j(x)$ for $x \in \Omega$ and $\varphi_j(x) = 0$ for $x \in \partial\Omega$. We begin this subsection by introducing a few properties of the eigenvalues of the operator Δ. For any $\tau \geq 0$, we also define the space

$$H^{\tau}(\Omega) = \Big\{ u \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\tau} \big|\langle u, \varphi_j \rangle\big|^2 < +\infty \Big\},$$

then $H^{\tau}(\Omega)$ is a Hilbert space endowed with the norm

$$\|u\|_{H^{\tau}(\Omega)} = \Big(\sum_{j=1}^{\infty} \lambda_j^{2\tau} |\langle u, \varphi_j \rangle|^2 \Big)^{\frac{1}{2}}.$$

Definition 2.1 ([11]). *Given $\xi \in H^{\mu}(\Omega)$ ($\mu > 0$) which have sequences of n (is known as **sample size**) discrete observations $\langle \xi^{\epsilon}, \varphi_j \rangle$, $j = 1, \ldots, n$. Non-parametric estimation of ξ is suggested as*

$$\xi_{n(\epsilon)} = \sum_{j=1}^{n} \langle \xi^{\epsilon}, \varphi_j \rangle \varphi_j(x). \tag{2.1}$$

Lemma 2.2. *Given $\xi \in H^{\tau}(\Omega)$ ($\tau > 0$), then the estimation errors are*

$$\mathbb{E}\|\xi_{n(\epsilon)} - \xi\|^2_{L^2(\Omega)} \leq \epsilon^2 n(\epsilon) + \frac{1}{\lambda_n^{2\tau}} ||\xi||^2_{H^{\tau}}. \tag{2.2}$$

Here $n(\epsilon) := n$ depends on ϵ and satisfies that $\lim_{\epsilon \to 0^+} n(\epsilon) = +\infty$.

Proof. Our proof starts with the observation that

$$\begin{aligned}
\mathbf{E}||\xi_{n(\epsilon)} - \xi||^2_{L^2(\Omega)} &= \mathbb{E}\left(\sum_{p=1}^{n} \langle \xi_{n(\epsilon)} - \xi, \varphi_j \rangle^2 \right) + \sum_{j=n+1}^{\infty} \langle \xi, \varphi_j \rangle^2 \\
&= \epsilon^2 \mathbb{E}\left(\sum_{j=1}^{n} W_j^2 \right) + \sum_{j=n+1}^{\infty} \lambda_j^{-2\tau} \lambda_j^{2\tau} \langle \xi, \varphi_j \rangle^2 \\
&\leq \epsilon^2 \mathbb{E}\left(\sum_{j=1}^{n} W_j^2 \right) + \frac{1}{\lambda_n^{2\tau}} \sum_{j=n+1}^{\infty} \lambda_j^{2\tau} \langle \xi, \varphi_j \rangle^2.
\end{aligned}$$

The assumption $W_j = \langle W, \varphi_j \rangle \overset{i.i.d}{\sim} N(0,1)$ implies that $\mathbb{E}W_j^2 = 1$. We then have the desired result. □

3 Recovering the source

In this subsection, we introduce the mild solution of the following initial value problem

$$\begin{cases} {}^c\partial_t^{\alpha} u(x,t) - \Delta u(x,t) = f(x), \quad x \in \Omega, x \in (0,T), \\ u(x,t) = 0, \quad x \in \partial\Omega, t \in (0,T], \\ u(x,0) = \rho(x), \quad x \in \Omega. \end{cases} \tag{3.1}$$

Using the separation of variables to yield the solution of (3.1). Suppose that the exact u is defined by Fourier series

$$u(x,t) = \sum_{j=1}^{\infty} u_j(t)\varphi_j(x), \text{ with } u_j(t) = \big\langle u(\cdot,t), \varphi_j \big\rangle. \tag{3.2}$$

From (3.2), we get

$$u_j(t) = \sum_{j=1}^{\infty} \Big[\exp\big(-\lambda_j t^\alpha \alpha^{-1} \big)\rho_j + \big\langle f, \varphi_j \big\rangle \int_0^t s^{\alpha-1} \exp\big(-\lambda_j \big(t^\alpha - s^\alpha\big)\alpha^{-1}\big) \mathrm{d}s \Big] \varphi_j(x).$$

Letting $t = T$ and $\rho_j = 0$, we get

$$\xi_j(x) = u_j(T) = \sum_{j=1}^{\infty} \Big[\big\langle f, \varphi_j \big\rangle \int_0^T s^{\alpha-1} \exp\big(-\lambda_j (T^\alpha - s^\alpha)\alpha^{-1}\big) \mathrm{d}s \Big] \varphi_j(x), \tag{3.3}$$

and then

$$f(x) = \sum_{j=1}^{\infty} \frac{\big\langle \xi, \varphi_j \big\rangle}{\int_0^T s^{\alpha-1} \exp\big(-\lambda_j (T^\alpha - s^\alpha)\alpha^{-1}\big) \mathrm{d}s}.$$

3.1 The ill-posedness

Theorem 3.1. *The inverse source problem is ill-posed.*

Proof. We will make the assumptions $\xi(x) = \varphi_k(x)$ $(1 \le k \le n)$, and the series of random observe values that follow the model $\langle \xi_\epsilon, \varphi_j \rangle = \langle \xi, \varphi_j \rangle + \epsilon \langle W, \varphi_j \rangle$, $(j = 1, \ldots, n)$. Estimation of the source function

$$\xi^k_{n(\epsilon)} = \varphi_k(x) + \sum_{j=1}^{n} \epsilon \langle W, \varphi_j \rangle \varphi_j(x)$$

and the source associate with random noise

$$f_{n(\epsilon)}(x) = \sum_{j=1}^{n} \frac{\big\langle \xi^k_{n(\epsilon)}, \varphi_j \big\rangle \varphi_j(x)}{\int_0^T s^{\alpha-1} \exp\big(-\lambda_j (T^\alpha - s^\alpha)\alpha^{-1}\big) \mathrm{d}s}.$$

We have

$$f_{n(\epsilon)}(x) - f(x) = \sum_{j=1}^{n} \frac{\epsilon \langle W, \varphi_j \rangle \varphi_j(x)}{\int_0^T s^{\alpha-1} \exp\big(-\lambda_j (T^\alpha - s^\alpha)\alpha^{-1}\big) \mathrm{d}s}$$

We have estimated

$$\mathbb{E} \| \xi^k_{n(\epsilon)} - \xi \|^2_{L^2(\Omega)} = \epsilon^2 n(\epsilon)$$

Estimates errors between $f^k(x)$ and $f(x)$ is given as follow

$$\begin{aligned}\mathbb{E}\|f_{n(\epsilon)}(x)-f(x)\|^2_{L^2(\Omega)} &= \sum_{j=1}^{n}\frac{\epsilon^2\mathbb{E}\langle W,\varphi_j\rangle^2}{\Big|\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)\mathrm{d}s\Big|^2}\\ &\geq \frac{\epsilon^2\mathbb{E}\langle W,\varphi_n\rangle^2}{\Big|\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_n(T^\alpha-s^\alpha)\alpha^{-1}\big)\mathrm{d}s\Big|^2}\end{aligned}$$

where

$$\Big|\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_n(T^\alpha-s^\alpha)\alpha^{-1}\big)\mathrm{d}s\Big|^2=\Big|\frac{1}{\lambda_n}\Big(1-\exp\big(-\lambda_nT^\alpha\alpha^{-1}\big)\Big)\Big|^2\leq\frac{1}{\lambda_n^2}$$

so, we have

$$\mathbb{E}\|f_{n(\epsilon)}(x)-f(x)\|^2_{L^2(\Omega)}\geq\lambda_n^2\epsilon^2.$$

In the case of $d=2$ by choosing $n(\epsilon)=1/\epsilon$, we have $\lim_{\epsilon\to0}\mathbb{E}\|\xi^k_{n(\epsilon)}-\xi\|^2_{L^2(\Omega)}=0$, however $\lim_{\epsilon\to0}\mathbb{E}\|f_{n(\epsilon)}(x)-f(x)\|^2_{L^2(\Omega)}=\infty$. We conclude that the inverse source problem is ill-posed. □

3.2 Conditional stability of the source

Theorem 3.2. *We have been working under the assumption that $f\in H^\tau(\Omega)$, one has*

$$\|f\|_{L^2(\Omega)}\leq C\|f\|^{\frac{2}{\tau+1}}_{H^\tau(\Omega)}\|\xi\|^{\frac{2\tau}{\tau+1}}_{L^2(\Omega)},$$

whereby

$$C=\big|1-\exp(-\lambda_1T^\alpha\alpha^{-1})\big|^{-2}.$$

Proof. Form now on, for a shorter,

$$Q(\lambda_j,\alpha)=\int\limits_0^T s^{\alpha-1}\exp\big(-\lambda_j(T^\alpha-s^\alpha)\alpha^{-1}\big)\mathrm{d}s,$$

by using the Hölder inequality, we have

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} &= \sum_{j=1}^\infty\Big|\frac{\langle\xi,\varphi_j\rangle}{Q(\lambda_j,\alpha)}\Big|^2=\sum_{j=1}^\infty\frac{\big|\langle\xi,\varphi_j\rangle\big|^{\frac{2}{\tau+1}}\big|\langle\xi,\varphi_j\rangle\big|^{\frac{2\tau}{\tau+1}}}{\big|Q(\lambda_j,\alpha)\big|^2}\\ &\leq\Big(\sum_{j=1}^\infty\frac{\big|\langle\xi,\varphi_j\rangle\big|^2}{\big|Q(\lambda_j,\alpha)\big|^{2\tau+2}}\Big)^{\frac{1}{\tau+1}}\Big(\sum_{j=1}^\infty\big|\langle\xi,\varphi_j\rangle\big|^2\Big)^{\frac{\tau}{\tau+1}}\\ &\leq\Big(\sum_{j=1}^\infty\frac{\big|\langle f,\varphi_j\rangle\big|^2}{\big|Q(\lambda_j,\alpha)\big|^2}\Big)^{\frac{1}{\tau+1}}\|\xi\|^{\frac{\tau}{\tau+1}}_{L^2(\Omega)}.\end{aligned}$$

Where

$$\left|Q(\lambda_j,\alpha)\right|^2 = \left|\frac{1}{\lambda_j}\Big(1-\exp\big(-\lambda_j T^\alpha \alpha^{-1}\big)\Big)\right|^2 \geq \left|\frac{1}{\lambda_j}\Big(1-\exp\big(-\lambda_1 T^\alpha \alpha^{-1}\big)\Big)\right|^2, \tag{3.4}$$

and this inequality leads to

$$\sum_{j=1}^{\infty}\frac{\left|\langle f,\varphi_j\rangle\right|^2}{\left|Q(\lambda_j,\alpha)\right|^{2\tau}} \leq \sum_{j=1}^{\infty}\frac{\lambda_j^{2\tau}\left|\langle f,\varphi_j\rangle\right|^2}{\left|1-\exp(-\lambda_1 T^\alpha\alpha^{-1})\right|^{2\tau}}. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\|f\|^2_{L^2(\Omega)} \leq \left|1-\exp(-\lambda_1 T^\alpha\alpha^{-1})\right|^{-2}\|f\|^{\frac{2}{\tau+1}}_{H^\tau(\Omega)}\|\xi\|^{\frac{2\tau}{\tau+1}}_{L^2(\Omega)}$$

by setting $C = \left|1-\exp(-\lambda_1 T^\alpha\alpha^{-1})\right|^{-2}$, then we get the result. □

4 Regularization

Theorem 4.1. *Given a positive constant $\tau > 0$. Assume that $\xi \in H^\tau(\Omega)$ and its statistics estimate is $\xi_{n(\epsilon)}$. The source fuction $f \in H^\tau(\Omega)$. If a regularized solution is given as follows*

$$f^\epsilon_{N(\epsilon)}(x) = \sum_{j=1}^{N(\epsilon)}\frac{\langle\xi_{n(\epsilon)},\varphi_j\rangle\varphi_j(x)}{|Q(\lambda_j,\alpha)|}, \tag{4.1}$$

then we have the estimation

$$\begin{aligned}\mathbb{E}\left\|f^\epsilon_{N(\epsilon)}-f\right\|^2_{L^2(\Omega)} \leq & \frac{2\lambda^{2\tau}_{N(\epsilon)}}{\left|1-\exp(-\lambda_1 T^\alpha\alpha^{-1})\right|^{2\tau}}\left(\epsilon^2 n(\epsilon)+\frac{1}{\lambda_n^{2\tau}}||\xi||^2_{H^\tau}\right)\\ &+2\big(N(\epsilon)\big)^{-2\tau}\left\|f\right\|^2_{H^\tau(\Omega)}\end{aligned}$$

where the regularization parameter $N(\epsilon)$ and the sample size $n(\epsilon)$ are chosen such that

$$\lim_{\epsilon\to 0^+} N(\epsilon) = +\infty, \quad \lim_{\epsilon\to0^+}\lambda^{2\tau}_{N(\epsilon)}\epsilon^2 n(\epsilon) = \lim_{\epsilon\to0^+}\frac{\lambda^{2\tau}_{N(\epsilon)}}{\lambda^{2\tau}_{n(\epsilon)}} = 0 \tag{4.2}$$

Remark 4.2. *There are many ways to choose the parameters $n(\epsilon)$ and $N(\epsilon)$ which could satisfy (4.2). Since $\lambda_{n(\epsilon)} \sim n(\epsilon)^{2/d}$, then one of the ways we can do by choosing the regularization parameter $N(\epsilon)$ such that $\lambda_{N(\epsilon)} = (n(\epsilon))^b$, where $0 < b < 4\tau/d$. The sample size $n(\epsilon)$ is chosen as $n(\epsilon) = (1/\epsilon)^{a/(b+1)}$, $0 < a < 2$.*

Proof. We have the truncation form of f

$$f_{N(\epsilon)}(x) = \sum_{j=1}^{N(\epsilon)}\frac{\langle\xi,\varphi_j\rangle\varphi_j(x)}{|Q(\lambda_j,\alpha)|}.$$

Using the triangle inequality, we get

$$\left\|f^\epsilon_{N(\epsilon)}-f\right\|_{L^2(\Omega)} \leq \left\|f^\epsilon_{N(\epsilon)}-f_{N(\epsilon)}\right\|_{L^2(\Omega)}+\left\|f_{N(\epsilon)}-f\right\|_{L^2(\Omega)}.$$

then

$$\mathbb{E}\big\|f^{\epsilon}_{N(\epsilon)} - f\big\|^2_{L^2(\Omega)} \leq 2\mathbb{E}\big\|f^{\epsilon}_{N(\epsilon)} - f_{N(\epsilon)}\big\|^2_{L^2(\Omega)} + 2\mathbb{E}\big\|f_{N(\epsilon)} - f\big\|^2_{L^2(\Omega)}. \tag{4.3}$$

In the following, we first consider the term $\mathbb{E}\big\|f^{\epsilon}_{N(\epsilon)} - f_{N(\epsilon)}\big\|^2_{L^2(\Omega)}$, we have

$$f^{\epsilon}_{N(\epsilon)}(x) - f_{N(\epsilon)}(x) = \sum_{j=1}^{\lambda_j \leq N(\epsilon)} \frac{\langle \xi^{\epsilon} - \xi, \varphi_j \rangle \varphi_j(x)}{Q(\lambda_j, \alpha)}$$

then

$$\begin{aligned}\big\|f^{\epsilon}_{N(\epsilon)} - f_{N(\epsilon)}\big\|^2_{L^2(\Omega)} &= \sum_{j=1}^{\lambda_j \leq N(\epsilon)} \frac{|\langle \xi^{\epsilon} - \xi, \varphi_j \rangle|^2}{|Q(\lambda_j, \alpha)|^2} \\ &\leq \sum_{j=1}^{\lambda_j \leq N(\epsilon)} \frac{\lambda_j^{2\tau}\big|\langle \xi^{\epsilon} - \xi, \varphi_j \rangle\big|^2}{\big|1 - \exp(-\lambda_1 T^{\alpha}\alpha^{-1})\big|^{2\tau}} \\ &\leq \frac{\lambda^{2\tau}_{N(\epsilon)}}{\big|1 - \exp(-\lambda_1 T^{\alpha}\alpha^{-1})\big|^{2\tau}} \|\xi_{n(\epsilon)} - \xi\|^2_{L^2(\Omega)}.\end{aligned}$$

Take expectation of both side

$$\mathbb{E}\big\|f^{\epsilon}_{N(\epsilon)} - f_{N(\epsilon)}\big\|^2_{L^2(\Omega)} \leq \frac{\lambda^{2\tau}_{N(\epsilon)}}{\big|1 - \exp(-\lambda_1 T^{\alpha}\alpha^{-1})\big|^{2\tau}} \left(\epsilon^2 n(\epsilon) + \frac{1}{\lambda_n^{2\tau}} ||\xi||^2_{H^{\tau}}\right) \tag{4.4}$$

Next, we continue to get the following estimate

$$\begin{aligned}\big\|f - f_{N(\epsilon)}\big\|^2_{L^2(\Omega)} &\leq \sum_{\lambda_j \geq N(\epsilon)}^{\infty} \lambda_j^{-2\tau}\lambda_j^{2\tau} \frac{\big|\langle \xi, \varphi_j \rangle\big|^2}{|Q(\lambda_j, \alpha)|^2} \leq \sum_{\lambda_j \geq N(\epsilon)}^{\infty} \lambda_j^{-2\tau}\lambda_j^{2\tau}\big|\langle f, \varphi_j \rangle\big|^2 \\ &\leq \lambda^{-2\tau}_{N(\epsilon)} \sum_{\lambda_j \geq N(\epsilon)}^{\infty} \lambda_j^{2\tau}\big|\langle f, \varphi_j \rangle\big|^2 \leq \lambda^{-2\tau}_{N(\epsilon)} \big\|f\big\|^2_{H^{\tau}(\Omega)}.\end{aligned} \tag{4.5}$$

Combining (4.3)–(4.5), the proof is completed. □

References

[1] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh (2014). *A new definition of fractional derivative*. Journal of Computational and Applied Mathematics, 264:65–70,. doi: 10.1016/j.cam.2014.01.002. URL https://doi.org/10.1016/j.cam.2014.01.002.

[2] Nguyen Huy Tuan, Erkan Nane, Donal O'Regan, and Nguyen Duc Phuong (2020). *Approximation of mild solutions of a semilinear fractional differential equation with random noise*. Proceedings of the American Mathematical Society, 148(8):3339–3357, April 2020a. doi: 10.1090/proc/15029. URL https://doi.org/10.1090/proc/15029.

[3] Peter Mathé and Sergei V. Pereverzev (2001). *Optimal discretization of inverse problems in hilbert scales. regularization and self-regularization of projection methods*. SIAM Journal on Numerical Analysis, *38*(6):1999–2021. doi: 10.1137/s003614299936175x. URL https://doi.org/10.1137/s003614299936175x.

[4] Dominique Picard and Gérard Kerkyacharian (2006). *Estimation in inverse problems and second-generation wavelets.* Proceedings of the International Congress of Mathematicians Madrid, pages 713–739. European Mathematical Society Publishing House. doi: 10.4171/022-3/37. URL https://doi.org/10.4171/022-3/37.

[5] Nguyen Anh Triet, Vo Van Au, Le Dinh Long, Dumitru Baleanu, and Nguyen Huy Tuan (2020). *Regularization of a terminal value problem for time fractional diffusion equation.* Mathematical Methods in the Applied Sciences, 43(6):3850–3878. doi: 10.1002/mma.6159. URL https://doi.org/10.1002/mma.6159.

[6] Fan Yang and Chu-Li Fu (2015). *The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation.* Applied Mathematical Modelling, *39*(5-6):1500–1512. doi: 10.1016/j.apm.2014.08.010. URL https://doi.org/10.1016%2Fj.apm.2014.08.010.

[7] Fan Yang, Yu-Peng Ren, Xiao-Xiao Li, and Dun-Gang Li (2017). *Landweber iterative method for identifying a space-dependent source for the time-fractional diffusion equation.* Boundary Value Problems, *2017*(1). doi: 10.1186/s13661-017-0898-2. URL https://doi.org/10.1186%2Fs13661-017-0898-2.

[8] Yaozong Han, Xiangtuan Xiong, and Xuemin Xue (2019). *A fractional landweber method for solving backward time-fractional diffusion problem.* Computers and Mathematics with Applications, *78*(1):81–91. doi: 10.1016/j.camwa.2019.02.017. URL https://doi.org/10.1016%2Fj.camwa.2019.02.017.

[9] Ha Vo Thi Thanh, Ngo Hung, and Nguyen Duc Phuong (2022). *Identifying inverse source for diffusion equation with conformable time derivative by fractional tikhonov method.* Advances in the Theory of Nonlinear Analysis and its Application, *6*(4):433 – 450. doi: 10.31197/atnaa.1079951.

[10] Nguyen Huy Tuan and Le Dinh Long (2017). *Fourier truncation method for an inverse source problem for space-time fractional diffusion equation.* Electron. J. Differential Equations, *2017*(122). URL http://ejde.math.txstate.edu.

[11] Nguyen Huy Tuan, Erkan Nane, Donal O'Regan, and Nguyen Duc Phuong (2020). *Approximation of mild solutions of a semilinear fractional differential equation with random noise.* Proceedings of the American Mathematical Society, *148*(8):3339–3357, April 2020b. doi: 10.1090/proc/15029. URL https://doi.org/10.1090/proc/15029.