Vo Viet Tri - Volume 4 - Issue 4-2022, p.1-17.

ĐẠI HỌC THỦ DẦU MỘT
2009 THU DAU MOT UNIVERSITY

Thu Dau Mot University
Journal of Science
ISSN 2615 - 9635
journal homepage: ejs.tdmu.edu.vn

Continuous dependence on parameters of second order differential inclusion and self-adjoint operator

by ***Vo Viet Tri*** *(Thu Dau Mot University)*

Article Info: *Received Sep. 6th, 2022, Accepted Nov. 9th,2022, Available online Dec. 15th, 2022*

Corresponding author: trivv@tdmu.edu.vn

https://doi.org/10.37550/tdmu.EJS/2022.04.356

ABSTRACT

In this paper, we establish compactness and continuous dependence on parameters for solution-set of the second order differential inclusion including self-adjoint operator in the form

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t),\mu), & (t,x) \in [0,T)\times\Omega \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, & x \in \Omega, \end{cases}$$

where $\mathcal{A}$ is a self-adjoint operator. We use the spectral theory on Hilbert spaces to obtain formulation for mild solutions. Using the mild solution formula together with a measure of noncompactness with values in an ordered space, we construct useful bounds for solution operators. Then, we establish necessarily upper semi-continuous and condensing settings, which mainly help to obtain the global existence of mild solutions and the compactness of the mild solution set. Finally, we provide a brief discussion on the continuous dependence of the solution-set on parameter μ.

Keywords: *multi-function, measure of compactness, differential inclusion, Self-Adjoint operator*

1 Introduction

Let Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega$ in Euclidean space $\mathbb{R}^N$ and T be a positive number. We first consider the following initial value problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t)), & (t,x) \in [0,T)\times\Omega, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial t^2}$ denote the symbols for the first and second-order derivatives with respect to the variable t, respectively; $\mathcal{A}$ is a self-adjoint operator on the Hilbert space $\mathcal{H}$, namely, $\langle \mathcal{A}^j u, w\rangle = \langle u, \mathcal{A}^j w\rangle$ for all $j \in \{1,2\}$; and F is a multi-valued mapping which is called source function.

Recently, differential equations and inclusions have gained much attention according to wide applications in economics, control theory, physics, etc, see e.g. [3, 7, 11, 18, 19]. There have been numerous studies on the existence and the stability of the solution of the problem with the source single-valued function or with non-integer order derivatives, in the literature [1, 2, 5–17].

In 2016, Anh et al. [1], studied the following fractional differential equation with a multi-valued source function

$$\partial_t^\alpha u(t) - Au(t) \in F(t,u,u_t), \quad t>0,\ 0<\alpha<1, \tag{1.2}$$

involving impulsive effects. They proved the global solvability and weakly asymptotic stability for solutions by analyzing the behavior of its solutions on the half-line. This equation was also studied in [5]. In [12], Phong-Lan concerned with the retarded fractional evolution inclusion

$$\partial_t^\alpha u(t) - Au(t) \in F(t,u_t), \quad t>0,\ 0<\alpha<1, \tag{1.3}$$

equipped with the history condition

$$u(s) = \varphi(s), \quad s \in [-h,0],\ h>0,$$

in a Banach space X, where A is a closed linear operator in X, F is a multi-map and φ is the history of solutions. Assuming F super-linear, they established the existence of decay global solution. However, in control theories, a common problem is that F is a multi-valued function. In addition to considering the existence and continuity of the solution set, the compactness of the solution set is also often of interest. In particular, when the input data F is noisy by the parameter μ, we need to consider the continuous dependence of the solution set on this parameter.

In [10], Ngoc and Tri discussed the existence and compactness of the solution set of following fractional pseudo-parabolic inclusion

$$\begin{cases} \partial_t^\alpha u + \kappa(-\Delta)^{\sigma_1}\partial_t^\alpha u + (-\Delta)^{\sigma_2} u & \in & F(t,u), & 0<t<T, \ x\in\Omega, \\ u(t,x) & = & 0, & 0<t<T, \ x\in\partial\Omega, \\ u(0,x) & = & \varphi(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where ∂_t^α represents the Caputo derivative over time of fractional order $\alpha\in(0,1)$. Using asymptotic behaviors of the Mittag-Leffler functions, the authors constructed useful bounds for the solution set to prove the compactness and continuous dependence on parameters of solutions set of Problem (1.4).

In [16], Tuan provided a regularized problem for bi-parabolic equation when the observed data are obtained in L^p ($p\neq 2$)

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = F(x,t,u(x,t)), \ \text{in } \Omega\times(0,T], \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_t = \frac{\partial u}{\partial t}$ and F is a single-valued function, in particular, $F(t,x) = \varphi(t)f(x)$ and Tuan introduced the error between the Fourier regularized solution and the exact solution in L^p spaces.

Our aim in this paper is devoted to study the initial value problem for differential inclusions (1.1). We establish the existence and the compactness of the solution set and discuss the dependence of the solutions of the following parameterized problems on the parameter μ in a metric space (E,d). It is more obvious that we consider the following problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\dfrac{\partial}{\partial t}u(t,x) + \mathcal{A}^2 u(t,x) \in F(t,u(t,x),\mu), \ (t,x)\in(0,T]\times\Omega \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, \qquad x\in\Omega, \end{cases} \tag{1.6}$$

In addition to widely used methods such as the evaluations by Fourier expansion of an element in the separable Hilbert space, the Gronwall's inequality, we use a measure of noncompactness ν in the ordered space generated by a convex cone to consider the existence of fixed points of the ν-condensing multi-map. To the extent of our knowledge, there are not many studies on differential inclusions containing self-adjoint operators with fractional order and techniques using the measure of noncompactness that takes values in cones.

Let $\mathcal{H}$ be a separable Hilbert, we denote by $Kv(\mathcal{H})$ (resp., $b(\mathcal{H})$) the all convex and compact (resp., bounded) subsets of $\mathcal{H}$ and consider problem (1.1) with the multifunction $F : [0,T] \times \mathcal{H} \to Kv(\mathcal{H})$ under the following conditions (H):

(Ha) for every $v \in \mathcal{H}$, the multimap $t \mapsto F(t,v)$ has a strongly measurable selection, i.e., there is a measurable function $f_v(.) : [0,T] \to \mathcal{H}$ satisfying $f_v(t) \in F(t,v)$;

(Hb) the multimap $F(t,.) : \mathcal{H} \to Kv(\mathcal{H})$ is upper semicontinuous (u.s.c) for a.e. $t \in [0,T]$;

(Hc) there exists a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u)||| := \sup_{v\in F(t,u)} \|v\|_{\mathcal{H}} \leq \alpha(t)(1+\|u\|_{\mathcal{H}}) \text{ for a.e. } t \in (0,T) \text{ and for all } u \in \mathcal{H};$$

(Hd) there is $B \in L^1((0,T);\mathbb{R})$ satisfying

$$\chi(F(t,D)) \leq B(t)\chi(D) \text{ for a.e. } t \in (0,T) \text{ for all } D \in b(\mathcal{H}),$$

here χ is MNC in $\mathcal{H}$ defined $\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}$.

Our work will be presented as follows. In the next section, we recall some basic properties of the multivalued analysis. Section 3 presents the global existence of mild solutions and compactness of the solution set of problem (1.1). Finally, we discuss on the continuous dependence parameters μ of the solution set of problems (1.6).

2 Preliminaries

Throughout this paper, let $\dot{\mathbb{N}} = \mathbb{N}\backslash\{0\}$ and $\mathscr{P}(E)$ (resp., $b(E), K(E)$) be the all nonempty (resp., bounded, compact) subsets of E. Let $\mathcal{H}$ be a separable Hilbert space with an inner product denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$. We denote by $\mathscr{C}([0,T];\mathcal{H})$ the space of all continuous functions from $[0,T]$ into $\mathcal{H}$ with norm $\|u\|_{\mathscr{C}([0,T],\mathcal{H})} = \sup_{t\in[0,T]} \|u(t,.)\|_{\mathcal{H}}$. The sequence $\{f_n\}$ in $\mathscr{C}([0,T];\mathcal{H})$ is said to be weakly convergent to f (resp., for almost every on $[0,T]$), written $f_n \rightharpoonup f$ (resp., a.e) on $[0,T]$), if $\langle f_n(t), f(t)\rangle$ tends to 0 for all (resp., a.e) $t \in [0,T]$. Let (E,ρ) be a metric space and $A \subset E$, we denote the distance between a point $x \in X$

and A by $\operatorname{dist}(x, A) := \inf\{\rho(x,y) : y \in A\}$, and the ϵ-neighbourhood of A by $\mathcal{N}_{\epsilon,\rho}(A) := \{y \in X : \operatorname{dist}(y, A) < \epsilon\}$ (in short, $\mathcal{N}_\epsilon(A)$).

To establish our main results we need some basic properties of multivalued analysis which can be found in [4]. Let us recall the concepts and these properties which shall be use in the next sections.

Definition 2.1. [4, Definition 2.1.1] Let E be a Banach space and $(C, \preceq)$ a partially ordered set. A map $\varphi : \mathcal{Y} \subset \mathscr{P}(E) \to C$ is said to be a *measure of noncompactness* (MNC) in $\mathcal{Y}$ if $\varphi(\overline{co}(D)) = \varphi(D)$ for all $D \in \mathcal{Y}$. A multi-mapping $F : E \to \mathcal{Y}$ is called *condensing* to φ (in short, φ-condensing) if $D \in \mathcal{Y}$ with $\varphi(D) \preceq \varphi(F(D))$ then D is relatively compact in E.

Let G be a subset of a metric space (E, d) and ϵ be a positive number. A subset A of E is said to be ϵ-net of G if $G \subset \bigcup_{x\in A} \{y \in E : d(x, y) < \epsilon\}$. If A is finite, A is called a *finite ϵ-net*. We need the Hausdorff measure χ which defined in [4, Definition 2.1.1], i.e., $\chi(G) = \inf\{\epsilon > 0 : G \text{ has a finite } \epsilon\text{-net}\}$.

Lemma 2.2. [4, Definition 2.1.1] *Let E be a Banach space and χ a Hausdorff MNC defined on family $\mathcal{F}$ of subsets of E. Then χ has the following properties:*

(a) monotone: if $D_1 \subset D_2$ implies $\chi(D_1) \le \chi(D_2)$, for all $D_1, D_2 \in \mathcal{F}$.

(b) algebraically semiadditive: if $\chi(D_1 + D_2) \le \chi(D_1) + \chi(D_2)$ for all $D_1, D_2 \in \mathcal{F}$.

(c) nonsingular: if $\chi(\{a\} \cup D) = \chi(D)$ for all $a \in E, D \in \mathcal{F}$.

(d) regular: $\chi(D) = 0$ if and only if D is relatively compact, $D \in \mathcal{F}$.

(e) semi-homogeneity: that is $\chi(\lambda D) = |\lambda|\chi(D)$ for all $\lambda \in \mathbb{R}, D \in \mathcal{F}$.

Definition 2.3. [4, Corollary 1.1.1] Let X and Y be topological spaces. A multimap $F : X \to \mathscr{P}(Y)$ is upper semicontinuous at the point $x \in X$ if, for every open set $W \subset Y$ such that $F(x) \subset W$, there exists a neighborhood $V(x)$ of x with property that $F(V(x)) \subset W$. A multimap is called *upper semicontinuous* (u.s.c) if it is upper continuous at every point $x \in X$.

When $(X, d), (Y, \rho)$ are metric spaces, it is clear that a multimap F form a metric space (X, d) into (Y, ρ) is u.s.c at point $x \in X$ iff for any $\epsilon > 0$, there exists $\delta > 0$ such that $F(w) \subset \mathcal{N}_{\epsilon,\rho}(F(x))$ for all $w \in \mathcal{N}_{\delta,d}(x)$.

For multimap $\mathcal{M} : E \to \mathscr{P}(E)$, we denote by $\operatorname{Fix}(\mathcal{M})$ the set of the all fixed points of $\mathcal{M}$, i.e., $\operatorname{Fix}(\mathcal{M}) = \{x \in E : x \in \mathcal{M}(x)\}$.

Lemma 2.4. [4, Corollary 3.3.1] *If M is a closed convex subset of Banach space E and $\mathcal{M} : M \to Kv(M)$ is a closed φ-condensing multimap, where φ is a nonsingular MNC defined on subsets of M, then* $\mathrm{Fix}(\mathcal{M}) \neq \emptyset$.

Lemma 2.5. [4, Propositions 3.5.1] *Let M be a closed subset of a Banach space E and $\mathcal{M} : M \to K(M)$ a closed multimap, which is φ-condensing on every bounded subset of M, where φ is a monotone MNC. If* $\mathrm{Fix}(\mathcal{M})$ *is bounded then it is compact.*

Lemma 2.6. [4, Propositions 3.5.2] *Let X be a closed subset of a Banach space E, β be a monotone MNC in E, Y be a metric space, and $G : Y \times X \to K(E)$ be a closed multimap which is β-condensing in the second variable and such that $F(\lambda) := \mathrm{Fix}\, G(\lambda, \cdot) \neq \emptyset$, for all $\lambda \in Y$. Then the multimap $F : Y \to \mathscr{P}(E)$ is u.s.c.*

Definition 2.7. ([4, Definition 4.2.1]) Let E be a Banach space. A $\{f_n\}_{n\in\mathbb{N}} \subset L^1([0,d],E)$ is called

1. *integrably bounded* if there is $\alpha \in L^1([0,d],\mathbb{R})$ such that

$$\|f_n(t)\|_E \leq \alpha(t) \ \text{ for a.e } t \in [0,d] \text{ for all } n \in \mathbb{N};$$

2. *semicompact* if it is integrably bounded and the set $\{f_n(t)\}_{n\in\mathbb{N}}$ is relatively compact for almost every $t \in [0,d]$.

In addition to the above mentioned basic properties of multivalued analysis, we also use the Gronwall's inequality presented in the following lemma.

Lemma 2.8. *(Gronwall) Let $a \geq 0$, $0 < T \leq \infty$, and continuous functions $\beta, \mu : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T \beta(s)ds < \infty$, and $\sup_{t\in[0,T]} \mu(t) < \infty$, $0 \leq \gamma \leq \xi \leq T$, and*

$$\mu(t) \leq a + \int_t^T \beta(s)\mu(s)ds \ \left(\textit{resp.}, \ \mu(t) \leq a + \int_0^t \beta(s)\mu(s)ds\right), \quad t \in [0,T].$$

Then $\mu(t) \leq ae^{\int_t^\xi \beta(s)ds}$ $\left(\textit{resp.}, \ \mu(t) \leq ae^{\int_\gamma^t \beta(s)ds}\right)$ for all $t \in [0,T]$.

3 Main results

In the first part of this section, we present the mild solution for problem (1.1). In the second part, we establish the existence and compactness of the solutions set. In the final part, on the basis of these results we discuss the continuous dependence of the solution set of the inclusion (1.6) on the parameter.

3.1 Mild solution

For $u \in \mathscr{C}([0,T];\mathcal{H})$, we denote

$$\mathcal{S}_F(u) = \left\{ f \in L^1((0,T);\mathcal{H}) | f(t,.) \in F(t,u), \text{ for a.e. } t \in (0,T) \right\}. \tag{3.1}$$

It is clear that $u = u(t,.)$ is a solution of Problem (1.1) if and only if there exists $f \in \mathcal{S}_F(u)$ satisfying

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) + 2\mathcal{A}\frac{\partial}{\partial t} u(t,x) + \mathcal{A}^2 u(t,x) = f(t,x), & (t,x) \in (0,T] \times \Omega \\ u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0, & x \in \Omega. \end{cases} \tag{3.2}$$

Assume that $\phi_\lambda \in \mathcal{H}$ is the eigen-function corresponding to the eigenvalue λ of the operator $\mathcal{A}$. Taking the inner product of both sides of (3.2) with ϕ_λ we obtain that

$$\frac{d^2}{dt^2}\langle u(t), \phi_\lambda\rangle + 2\lambda \frac{d}{dt}\langle u(t), \phi_\lambda\rangle + \lambda^2 \langle u(t), \phi_\lambda\rangle = \langle f(t), \phi_\lambda\rangle. \tag{3.3}$$

By the method of constant variation, from (3.3) it follows that

$$\langle u(t), \phi_\lambda\rangle = \int_0^t (t-s)\langle f(s), \phi_\lambda\rangle e^{-\lambda(t-s)} ds. \tag{3.4}$$

Throughout this paper, let ϕ_n, $n \in \dot{\mathbb{N}}$, be the eigenfunction corresponding to the eigenvalues λ_n satisfying $0 < \lambda_1 < \lambda_2 < ...$, and $\lim_{n\to\infty} \lambda_n = \infty$. Furthermore, assume that $\{\phi_n\}_{n\in\dot{\mathbb{N}}}$ is an orthonormal basis of $\mathcal{H}$. If Problem (3.2) has a solution $u \in \mathscr{C}([0,T],\mathcal{H})$, then

$$u(t) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle f(s), \phi_n\rangle \phi_n e^{-\lambda_n(t-s)} ds. \tag{3.5}$$

This suggests to define the mild solution of the problem (1.1) as follows:

Definition 3.1. A function $u \in \mathscr{C}([0,T];\mathcal{H})$ is said to be a mild solution of Problem (1.1) if the following conditions are fulfilled

(i) $u(0,\cdot) = \frac{\partial}{\partial t} u(0,\cdot) = 0$, and

(ii) there exists $f \in \mathcal{S}_F(u)$ such that for every $t \in [0,T]$,

$$u(t,.) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle f(s), \phi_n\rangle \phi_n(.) e^{-\lambda_n(t-s)} ds. \tag{3.6}$$

Since $f \in L^1((0,T);\mathcal{H})$, it is clear that (3.6) is well defined and $u(t,.) \in \mathcal{H}$ for a.e $t \in [0,T]$.

3.2 Upper semicontinuous and condensing settings

For $f \in L^1((0,T);\mathcal{H})$ we define

$$\Phi(f)(t,.) = \sum_{n=1}^{\infty} \int_0^t (t-s)\langle f(s), \phi_n\rangle \phi_n(.)e^{-\lambda_n(t-s)}ds. \tag{3.7}$$

It is clear that Φ is well defined. In this subsection, we establish the u.s.c and χ-condensing properties of the multimap $\Phi \circ \mathcal{S}_F$.

Lemma 3.2. *Let $\{f_n\} \subset L^1((0,T);\mathcal{H})$ be a semicompact sequence. Then, the following statements hold.*

a) The set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is equicontinuous.

b) The set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relatively compact in $\mathscr{C}([0,T];\mathcal{H})$.

c) $\Phi(f_n) \to \Phi(f_0)$ if $f_n \rightharpoonup f_0$.

Proof. We first begin with proving the assertion a). Assume that $t, t' \in [0,T]$ satisfying $0 \le t < t' \le T$. We write

$$\Phi(f_n)(t) - \Phi(f_n)(t') = \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t) - \sum_{j=1}^{\infty} \mathcal{R}_j(n)(t'), \tag{3.8}$$

here

$$\mathcal{R}_j(n)(t) = \int_0^t \alpha_n(t,s,j)ds, \quad \alpha_n(t,s,j) = (t-s)\langle f_n(s), \phi_j\rangle \phi_j e^{-\lambda_j(t-s)}.$$

Then, we get

$$\mathcal{R}_j(n)(t) - \mathcal{R}_j(n)(t') = \int_0^t (\alpha_n(t,s,j) - \alpha_n(t',s,j))\, ds - \int_t^{t'} \alpha_n(t',s,j)ds. \tag{3.9}$$

Using the mean value theorem for function $t \mapsto (t-s)e^{-\lambda_j(t-s)}$, we obtain

$(t-s)e^{-\lambda_j(t-s)} - (t'-s)e^{-\lambda_j(t'-s)} = (1-\lambda_j(\xi_j - s))e^{-\lambda_j(\xi_j-s)}(t-t')$ for some $\xi_j \in (t,t')$.

Therefore, from the condition $0 \le s \le t < \xi_j \le t' \le T$ it implies that the set $\{\mu_j : j = 1,2,...\}$, here $\mu_j = (1-\lambda_j(\xi_j - s))^2 e^{-2\lambda(\xi_j - s)}$, is bounded. Hence,

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}(\alpha_n(t,s,j) - \alpha_n(t',s,j))\right\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty}(1-\lambda_j(\xi_j-s))^2\langle f_n(s),\phi_j\rangle^2 e^{-2\lambda_j(\xi_j-s)}|t'-t|^2 \\ &\le C_1 \sum_{j=1}^{\infty}\langle f_n(s),\phi_j\rangle^2 |t'-t|^2 \\ &= C_1\|f_n(s)\|_{\mathcal{H}}^2 |t'-t|^2. \end{aligned} \tag{3.10}$$

Further, we have

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}\alpha_n(t',s,j)\right\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty}(t-s)^2\langle f_n(s),\phi_j\rangle^2\\ &\le C_2\sum_{j=1}^{\infty}\langle f_n(s),\phi_j\rangle^2\\ &= C_2\|f_n(s)\|_{\mathcal{H}}^2. \qquad (3.11)\end{aligned}$$

Combination of (3.11), (3.10), (3.9) and (3.8) it show that

$$\|\Phi(f_n)(t)-\Phi(f_n)(t')\|_{\mathcal{H}} \le \sqrt{C_2}\int_t^{t'}\|f_n(s)\|_{\mathcal{H}}ds + \sqrt{C_1}\int_0^t\|f_n(s)\|_{\mathcal{H}}ds|t'-t| \qquad (3.12)$$

Since the sequence $\{f_n\}$ is integrably bounded, there exists $\alpha \in L^1([0,T],\mathbb{R})$ such that $\|f_n(s)\|_{\mathcal{H}} \le \alpha(s)$ for a.e $s \in [0,T]$ and for all $n \in \dot{\mathbb{N}}$. From (3.12) we evaluate

$$\|\Phi(f_n)(t)-\Phi(f_n)(t')\|_{\mathcal{H}} \le C|t'-t| \quad \text{for all } n=1,2,... \qquad (3.13)$$

This deduce the assertion a).

Next, we will prove the set $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is bounded at any point $t \in [0,T]$. Indeed, for every $t \in [0,T]$, since $\{f_n\}$ is integrally bounded we get

$$\begin{aligned}\|\Phi(f_n)(t)\|_{\mathcal{H}} &\le C_0\int_0^T\|f_n(s)\|_{\mathcal{H}}ds\\ &\le C_0\int_0^T\alpha(s)ds = C \ \ \forall n \in \dot{\mathbb{N}}. \qquad (3.14)\end{aligned}$$

By Arzela-Ascoli theorem, it implies that $\{\Phi(f_n) : n \in \dot{\mathbb{N}}\}$ is relative compact in $\mathscr{C}([0,T],\mathcal{H})$. Assertion c. is a consequence of b. with the note that Φ is bounded linear mapping from $L^1((0,T);\mathcal{H})$ to $\mathscr{C}([0,T];\mathcal{H})$. □

Using the upper semicontinuous assumption (Hb) of F and applying Mazur's theorem, we obtain the following lemma.

Lemma 3.3. *Let* $\{v_n\}_{n\in\dot{\mathbb{N}}} \subset \mathscr{C}([0,T];\mathcal{H})$ *and* $\{f_n\}_{\in\dot{\mathbb{N}}} \subset L^1((0,T);\mathcal{H})$ *satisfying* $f_n \in \mathcal{S}_F(v_n)$ *for all* $n \ge 1$. *Then, if* $v_n \to v$ *and* $f_n \rightharpoonup f$, $f \in \mathcal{S}_F(v)$.

The closed property of the multioperator $\Phi \circ \mathcal{S}_F$ which consequence of the use Lemma 3.2 and Lemma 3.3.

Lemma 3.4. *Assume that the condition (H) is satisfied. Then* $\Phi \circ \mathcal{S}_F$ *is closed multioperators from* $L^1((0,T);\mathcal{H})$ *into* $\mathscr{C}([0,T],\mathcal{H})$.

Proof. We prove the closed property of $\Phi \circ \mathcal{S}_F$, the one is argued similarly for $\Psi \circ \mathcal{S}_F$. Assume that sequences $\{v_n\}_{n\geq 1}$ and $\{z_n\}_{n\geq 1}$ in $\mathscr{C}([0,T];\mathcal{H})$ satisfying

$$v_n \to v, \;\; z_n \in \Phi \circ \mathcal{S}_F(v_n) \text{ and } z_n \to z.$$

We will show that $z \in \Phi \circ \mathcal{S}_F(v)$. Indeed, let $\{f_n\}$ be an arbitrary sequence in $L^1((0,T);\mathcal{H})$ satisfying $f_n \in \mathcal{S}_F(v_n)$ and $z_n = \Phi(f_n)$. From the condition (Hc) it follows that $\{f_n\}$ is integrally bounded. Further, from the condition (Hd) it follows that $\{f_n\}$ is semicompact and also weakly compact in $L^1((0,T);\mathcal{H})$ (see [4, Theorem 5.1.2]). Without loss of generality, we may assume that $f_n \rightharpoonup f \in L^1((0,T);\mathcal{H})$. Using Lemma 3.2, we get $\Phi(f_n) \to \Phi(f) = z$, so by Lemma 3.3 we deduce $z \in \Phi \circ \mathcal{S}_F(v)$. □

The following lemma is a consequence of Lemma 3.2 and Lemma 3.4.

Lemma 3.5. *Assume that the condition (H) is fulfilled. Then, the multioperator* $\Phi \circ \mathcal{S}_F$ *is u.s.c.*

Next, we present the condensing property of the multioperator $\Phi \circ \mathcal{S}_F$ associated with a suitable measure of noncompactness. Let $D \subset \mathscr{C}([0,T],\mathcal{H})$, we denote by $\Delta(D)$ the family of all denumerable subsets of D. Let L be a positive constant, we define

$$\nu_L(D) \triangleq \max_{Q\in\Delta(D)} \big(\gamma_L(Q);\, \mathrm{mod}_{\mathrm{C}}(Q)\big),$$

where

$$\gamma_L(Q) \triangleq \sup_{t\in[0,T]} e^{Lt}\chi(Q(t)), \quad \mathrm{mod}_{\mathrm{C}}(Q) \triangleq \lim_{\delta\to 0}\sup_{v\in D}\max_{|t'-t|\leq\delta} \|v(t') - v(t)\|,$$

$Q(t) = \{w(t) : w \in Q\}$. The MNC ν_L has the all properties which presented in Lemma 2.2. The reader can find their proofs in [4, Example 2.1.4].

Lemma 3.6. *Assume that (H) is satisfied,* $\mathcal{S}_F : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(L^1(0,T);\mathcal{H})$ *defined by* (3.1) *and* Φ *given by* (3.7). *Then, there exists* $L > 0$ *such that* $\Phi \circ \mathcal{S}_F$ *is* ν_L*-condensing.*

Proof. Let D be a bounded subset of $\mathscr{C}([0,T];\mathcal{H})$ satisfying

$$\nu_L(D) \preceq \nu_L(\Phi \circ \mathcal{S}_F), \tag{3.15}$$

here the order $\preceq$ is taken in $\mathbb{R}^2$ induced by the positive cone $\mathbb{R}_+ \times \mathbb{R}_+$. We will prove that D is relatively compact. Let $\{v_n\}$ be any sequence in D, we set $g_n(t,.) = \Phi(f_n)(t,.)$ with $f_n \in \mathcal{S}_F(v_n)$ and

$$\nu_L(\{g_n : n \geq 1\}) = (\gamma_L(\{g_n : n \geq 1\}); \text{mod}_C(\{g_n : n \geq 1\}))\,,$$

the number L will be determined later. We have

$$\begin{aligned}
& e^{-Lt}\chi(\{g_n(t,.) : n \geq 1\}) \\
& = e^{-Lt}\chi\left(\left\{\sum_{j=1}^{\infty}\left(\int_0^t (t-s)\langle f_n(s), \phi_j\rangle e^{-\lambda(t-s)}ds\right)\phi_j(.) : n \geq 1\right\}\right) \\
& \leq C_0 e^{-Lt}\int_0^t \chi\left(\{f_n(s) : n \geq 1\}\right)ds \\
& \leq C_1 \sup_{s\in[0,T]}\left(e^{-Ls}\chi(\{v_n(s,.) : n \geq 1\})\right)\int_0^t s^{\gamma_1}e^{-L(t-s)}ds, \qquad (3.16)
\end{aligned}$$

where we have used χ-regularity condition (Hd) in the last estimate. From the above inequality we obtain

$$\gamma_L(\{g_n : n \geq 1\}) \leq C_1\left(\sup_{t\in[0,T]}\int_0^t s^{\gamma_1}e^{-L(t-s)}ds\right)\gamma_L(\{v_n : n \geq 1\}). \qquad (3.17)$$

Since

$$\lim_{L\to\infty}\left(\sup_{t\in[0,T]}\int_0^t s^{\gamma}e^{-L(t-s)}ds\right) = 0, \quad (\gamma > -1),$$

there exists $L_0 > 0$ such that

$$\sup_{t\in[0,T]}\int_0^t s^{\gamma_1}e^{-L(t-s)}ds < \frac{1}{4C_1} \quad \text{for all } L \geq L_0. \qquad (3.18)$$

On the other hand, from (3.15) it implies $\gamma_{L_0}(\{g_n : n \geq 1\}) \geq \gamma_{L_0}(\{v_n : n \geq 1\})$. Hence, combining with (3.17) and (3.18) we get $\gamma_{L_0}(\{v_n : n \geq 1\}) = 0$. So $\chi(\{v_n(t,.)\}) = 0$ for all $t \in [0,T]$. From the conditions (Hc) and (Hd) it implies that $\{f_n\}$ is semicompact. Applying Lemma 3.2 we deduce that $\{g_n : n \geq 1\}$ is relatively compact, so $\nu_{L_0}(D) = (0,0)$. The proof is completed. □

3.3 Existence and compactness

In this subsection, we shall establish the compact property of the mild solutions set, denoted by $\mathscr{S}_h^F[0,T]$, of the inclusion (1.1).

Theorem 3.7. *Assume that F satistied the condition (H). Then, $\mathscr{S}_h^F[0,T]$ is a nonempty and compact subset of $\mathscr{C}([0,T];\mathcal{H})$.*

Proof. We consider the multimap $\mathcal{M}:\mathscr{C}([0,T];\mathcal{H})\to\mathscr{P}(\mathscr{C}([0,T];\mathcal{H}))$ defined by

$$\mathcal{M}(u):=\{v\in\mathscr{C}([0,T];\mathcal{H}):v(t,.)=\Phi(f)(t,.),f\in\mathcal{S}_F(u)\}$$

Choose C_1 satisfying

$$\|\Phi(f)(t)\|_{\mathcal{H}}\leq C_1\int_0^t\|f(s)\|_{\mathcal{H}}ds. \tag{3.19}$$

and L_0 satisfying (3.18). Applying Lemma 3.5 and Lemma 3.6 we derive that $\mathcal{M}$ is u.s.c and ν_{L_0}-condensing. We define the weighted space

$$\mathscr{C}_{L_0}([0,T];\mathcal{H})=\left\{v\in\mathscr{C}([0,T];\mathcal{H}):\exists K>0,\|v(t,.)\|_{\mathcal{H}}\leq Ke^{L_0t}\ \forall t\in[0,T]\right\},$$

endowed with norm

$$\|v\|_{\mathscr{C}_{L_0}([0,T];\mathcal{H})}=\sup_{t\in[0,T]}e^{-L_0t}\|v(t,\cdot)\|_{\mathcal{H}}\quad\forall v\in\mathscr{C}_{L_0}([0,T];\mathcal{H}).$$

In this space, we denote by $\overline{B}(r)$ the closed ball centered at the zero function with the radius r. Choose $r>(r+1)/4$. Let $u\in\overline{B}(r)$, $f\in\mathcal{S}_F(u)$, $v\in\mathcal{M}(u)$. Using the condition (Hc) we have

$$\begin{aligned}e^{-L_0t}\|v(t,.)\|_{\mathcal{H}}&=e^{-L_0t}\,\|\Phi(f)(t,.)\|_{\mathcal{H}}\\&\leq C_1\int_0^t e^{-L_0(t-s)}e^{-L_0s}s^{\gamma_1}\left(1+\|u(s,.)\|_{\mathcal{H}}\right)ds\\&\leq C_1\int_0^t s^{\gamma_1}(e^{-L_0s}+r)e^{-L_0(t-s)}ds\\&\leq C_1\left((1+r)\int_0^t s^{\gamma_1}e^{-L_0(t-s)}ds\right)<r.\end{aligned}$$

This implies $v\in\overline{B}(r)$. It follows that $\mathscr{S}_h^F[0,T]\neq\emptyset$ by applying Lemma 2.4. To prove that $\mathscr{S}_h^F[0,T]$ is compact. This is argued similarly to the last part in the proof of the previous theorem. □

3.4 Continuous dependence on parameters

In this subsection, we consider the dependence of the solution of the following parameterized problem (1.6) on the parameter μ in a metric space (E,d). For convenience, we recall the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) + 2\mathcal{A}\frac{\partial}{\partial t}u(t,x) + \mathcal{A}^2u(t,x) \in F(t,u(t),\mu), \ (t,x) \in (0,T] \times \Omega, \\ u(0,x) = \frac{\partial}{\partial t}u(0,x) = 0, \qquad x \in \Omega. \end{cases} \tag{3.20}$$

For a given $\mu_0 \in E$, we consider the continuous of respectively mild solutions set, namely, if μ near enough μ_0, the solution set corresponding to μ is contained in neighbourhood of the solution sets corresponding to μ_0.

We establish the continuous dependence on parameters with the following assumptions (H_μ) on nonlinearity.
Let $F : [0,T] \times \mathcal{H} \times E \to Kv(\mathcal{H})$ be a multimap satisfying the following conditions:

H_μ(a) : The multimap $F(.,u,\mu)$ has a strongly measurable selection for every $(u,\mu) \in \mathcal{H} \times E$;

H_μ(b) : The multimap $F(t,.,.) : \mathcal{H} \times E \to Kv(\mathcal{H})$ is u.s.c for a.e. $t \in [0,T]$;

H_μ(c) : There is a function $\alpha \in L^1((0,T);\mathbb{R})$ such that

$$|||F(t,u,\mu)||| := \sup_{v \in F(t,u,\mu)} \|v\|_{\mathcal{H}} \leq \alpha(t)(1 + \|u\|_{\mathcal{H}}) \text{ for a.e. } t \in (0,T),$$

for all $u \in \mathcal{H}, \mu \in E$.

H_μ(d) : There exists $\mathcal{B} \in L^1((0,T);\mathbb{R})$ satisfying

$$\chi(F(t,G,E)) \leq \mathcal{B}(t)\chi(G) \text{ for a.e. } t \in (0,T) \text{ for all } G \in b(\mathcal{H}),$$

here χ is MNC in $\mathcal{H}$ defined

$$\chi(G) = \inf\{\varepsilon > 0 : G \text{ has a finite } \varepsilon\text{-net}\}. \tag{3.21}$$

For $(u,\mu) \in \mathscr{C}([0,T];\mathcal{H}) \times E$ we denote

$$\mathcal{S}_{F,\mu}(u) = \left\{f \in L^1((0,T);\mathcal{H}) | f(t,.) \in F(t,u,\mu), \text{ for a.e. } t \in (0,T)\right\}$$

For every $\mu \in E$, similarly as Theorem 3.7 we also denote multioperator $\mathcal{M}_\mu : \mathscr{C}([0,T];\mathcal{H}) \to \mathscr{P}(\mathscr{C}([0,T];\mathcal{H}))$ defined by

$$\mathcal{M}_\mu(u) := \{v \in \mathscr{C}([0,T];\mathcal{H}) : v(t,.) = \Phi(f)(t,.), f \in \mathcal{S}_{F,\mu}(u)\}.$$

Denote by $\mathscr{H}_h^{F,\mu}$ the family of all local mild solutions of Problem (1.6), i.e. , $u \in \mathscr{H}_h^{F,\mu}$ iff there exist $\tau \in (0,T]$ and $u \in \mathscr{C}([0,T];\mathcal{H})$ such that for all $\overline{\tau} \in [0,\tau]$ and $v_{\overline{\tau}} = u|_{[0,\overline{\tau}]}$, it holds

$$v_{\overline{\tau}} \in \{w \in \mathscr{C}([0,\overline{\tau}];\mathcal{H}) : w(t) = \Phi(f)(t), f \in \mathcal{S}_{F,\mu}(u)\},$$

and $\mathscr{H}_h^{F,\mu}[0,T] := \left\{v \in \mathscr{H}_h^{F,\mu} : v \in \mathcal{M}_\mu(v)\right\}$, here

$$\mathcal{M}_\mu(u) := \{v \in \mathscr{C}([0,T];\mathcal{H}) : v(t) = \Phi(f)(t), f \in \mathcal{S}_{F,\mu}(u)\}.$$

Theorem 3.8. *Assume that the condition* (H_μ) *holds, for some* $\mu_0 \in E$ *the set* $\mathscr{H}_h^{F,\mu_0}[0,T]$ *is bounded and*

$$\mathscr{H}_h^{F,\mu_0}[0,\overline{\tau}] = \mathscr{H}_h^{F,\mu_0}[0,T]|_{[0,\overline{\tau})} \quad \text{for all } \overline{\tau} \in (0,T]. \tag{3.22}$$

Then, for any $\epsilon > 0$*, there exists* $\delta_\epsilon > 0$ *satisfying*

$$\mathscr{H}_h^{F,\mu}[0,T] \subset \mathcal{N}_\epsilon\left(\mathscr{H}_h^{F,\mu_0}[0,T]\right) \quad \text{for all } \lambda \in \mathcal{B}_{\delta_\epsilon}(\mu_0).$$

Proof. Suppose that $r > 0$ with $|||\mathscr{H}_h^{F,\mu}[0,T]||| < r$. We first will prove the following statement by contraction argument: There is $\delta > 0$ such that $\mu \in \mathcal{N}_\delta(\mu_0) \subset E$ implies

$$|||\mathscr{H}_h^{F,\mu}(t)||| \le 3r \quad \text{for all } t \in [0,T]. \tag{3.23}$$

Indeed, we assume by contradiction that (3.23) fails. Then, we can take sequences $\{\mu_n\} \subset E$, $\{t_n\} \subset [0,T]$, $\{u_n\} \subset \mathscr{C}([0,T];\mathcal{H})$, $\mu_n \to \mu_0$ such that $w_n \in \mathcal{M}_{\mu_n}(w_n)$ and

$$\operatorname{dist}\left(w_n(t_n), \mathscr{H}_h^{F,\mu_0}(t_n)\right) \ge 2r, \quad \operatorname{dist}\left(w_n(t), \mathscr{H}_h^{F,\mu_0}(t)\right) < 2r \tag{3.24}$$

for all $t \in [0,t_n)$.

Denote $t_* = \underline{\lim}\{t_n\}$ we shall prove that $t_* \in (0,T]$. Indeed, assume that $t_* = 0$. Let us choose a sub-sequence of $\{t_n\}$ converging to 0, which we also denote by $\{t_n\}$ for convenience. Since $\mathscr{H}_h^{F,\mu_0}$ is bounded and from (3.22) it follows that $\mathscr{H}_h^{F,\mu_0}$ is compact, and so the distance between h and $\mathscr{H}_h^{F,\mu_0}(t_n)$ tends to zero. It observes that

$$\begin{aligned} 2r &\le \operatorname{dist}\left(w_n(t_n), \mathscr{H}_h^{F,\mu_0}(t_n)\right) \\ &\le \|w_n(t_n) - h\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathscr{H}_h^{F,\mu_0}(t_n)\right) \\ &\le \left\|\sum_{j=1}^{\infty} e^{\mu_j(T-t_n)}\langle h,\phi_j\rangle\phi_j - h\right\|_{\mathcal{H}} + \|\Phi(f_n)(t_n)\|_{\mathcal{H}} + \operatorname{dist}\left(h, \mathscr{H}_h^{F,\mu_0}(t_n)\right), \end{aligned} \tag{3.25}$$

here $f_n \in \mathcal{S}_{F,\mu_0}(w_n)$ for all $n = 1, 2, \ldots$. Letting $n \to \infty$ in (3.25) we derive the contradiction $2r \leq 0$. Summarily, we deduce $t_* > 0$.

By the definition of t_*, there exists a number γ with $0 < \gamma < t_* \leq T$ such that all solutions w_n are defined on $[0, t_* - \gamma]$. We next prove that for every w_n, there exists $\tau_n \in [0, t_* - \gamma] \subsetneq [0, T]$ satisfying

$$\operatorname{dist}\big(w_n(\tau_n), \mathscr{H}_h^{F,\mu_0}(\tau_n)\big) \geq \epsilon. \tag{3.26}$$

For every n, let any $t_\dagger \in [0, t_n)$, by the compactness of $\mathscr{H}_h^{F,\mu_0}$ we can assume that $\|w_n(t) - w_\dagger(t)\|_{\mathcal{H}} < \epsilon$ for some $w_\dagger \in \mathscr{H}_h^{F,\mu_0}$. Then

$$\begin{aligned} &\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathcal{H}} \\ &\qquad \leq \|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathcal{H}} + \|w_\dagger(t_\dagger + t) - w_\dagger(t_\dagger)\|_{\mathcal{H}} + \|w_n(t_\dagger) - w_\dagger(t_\dagger)\|_{\mathcal{H}}. \end{aligned}$$

With the same arguments as the first part of Lemma 3.2, one can choose t small enough such that both $\|w_n(t_\dagger + t) - w_n(t_\dagger)\|_{\mathcal{H}}$ and $\|w_\dagger(t_\dagger + t) - w_\dagger(t_\dagger)\|_{\mathcal{H}}$ are less than $\epsilon/4$. Hence, the norms $\|w_n(t_\dagger + t) - w_\dagger(t_\dagger + t)\|_{\mathcal{H}} \leq 3\epsilon/2$, which contradicts (3.24). Namely, (3.26) is proved.

Now, by similar arguments as obtaining Lemma 3.6, we note that the multimap $\mathcal{M}_* : E \times \mathscr{C}([0, t_* - \gamma]; \mathcal{H}) \to Kv(C([0, t_* - \gamma]; \mathcal{H}))$, $\mathcal{M}_*(\mu, u) = \mathcal{M}^\mu(u)$, is ν_L-condensing for some $L > 0$. This ensures the relative compactness of the sequence $\{w_n|_{[0,t_*-\gamma]}\}$. Let us take $w_* = \lim w_n|_{[0,\gamma-t_*]}$, which belongs to $\mathcal{M}_*(\lambda_0, w_*)$ on $[0, t_* - \gamma]$. Thus, by passing to the limit in (3.26) we obtain

$$\operatorname{dist}\big(w_*(t_*), \mathscr{H}_h^{F,\mu_0}(t_*)\big) \geq \epsilon.$$

Consequently, the solution u_* cannot be extended to the interval $[0, T]$, which contradicts (3.22). Finally, we complete the proof by applying Lemma 2.6. □

4 Acknowledgements

This research is funded by Thu Dau Mot University, Binh Duong Province, Vietnam under grant number DT.21.1-014.

References

[1] Thanh Anh, N., Dinh Ke, T. & Nhu Quan, N. (2016). *Weak stability for integro-differential inclusions of diffusion-wave type involving infinite delays*, Discrete Contin. Dyn. Syst. Ser. B, *21*(10), 3637–3654.

[2] Caraballo, T., Bao Ngoc,T., Huy Tuan, N. & Wang, R. (2021). *On a nonlinear Volterra integrodifferential equation involving fractional derivative with Mittag-Leffler kernel*, Proceedings of the American Mathematical Society, 149, 3317-3334.

[3] Van Hung, N. & Viet Tri, V. (2020). *Stability analysis for parametric symmetric vector quasi-equilibrium problems with application to traffic network problems*, J. of Nonlinear and Convex Analysis, *21*(10), 2207–2223.

[4] Kamenskii, M., Obukhovskii, V. & Zecca, P. (2001). Condensing multivalued maps and semilinear differential inclusions in Banach spaces, De Gruyter Series in Nonlinear Analysis and Applications, 7. Walter de Gruyter & Co., Berlin.

[5] Dinh Ke, T. &Lan, D. (2017). *Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects*, J. Fixed Point Theory Appl., *19*(4), 2185–2208.

[6] Kian, Y., Oksanen, L., Soccorsi, E. & Yamamoto, M. (2018). *Global uniqueness in an inverse problem for time fractional diffusion equations*, J. Differ. Equ., 264, 1146–1170.

[7] Kiryakova, V. (1994). Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics 301, Longman, Harlow.

[8] Bao Ngoc, T., Caraballo, T., Huy Tuan, N. & Zhou, Y. (2021). *Existence and regularity results for terminal value problem for nonlinear fractional wave equations*, Nonlinearity, *34*, 55.

[9] Bao Ngoc, T., Viet Tri, V., Hammouch, Z. & Huu Can, N. (2021). *Stability of a class of problems for time-space fractional pseudo-parabolic equation with datum measured at terminal time*, Appl. Numer. Math., *167*, 308-329.

[10] Bao Ngoc, T. & Viet Tri, V. (2022). *Global existence and continuous dependence on parameters for space-time fractional pseudo-parabolic inclusion*, J. Nonlinear and Convex Analysis, *23*(7), 1469-1485.

[11] Anh Tuan, N., Hammouch, Z., Karapinar, E. & Huy Tuan, N. (2021). *On a nonlocal problem for a Caputo time-fractional pseudoparabolic equation*, Math Meth Appl Sci., DOI: 10.1002/mma.7743

[12] Nam Phong, V. & Lan, D. (2021) . *Finite-time attractivity of solutions for a class of fractional differential inclusions with finite delay*, J. Pseudo-Differ. Oper. Appl., *12*(5), https://doi.org/10.1007/s11868-021-00374-2

[13] Sakamoto, K. & Yamamoto, M. (2011). *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl., *382*(1), 426–447.

[14] Huy Tuan, N., Bao Ngoc, T., Zhou, Y. & O'Regan, D. (2020). *On existence and regularity of a terminal value problem for the time fractional diffusion equation*, Inverse Problems, *36*(5), 055011.

[15] Huy Tuan, N., Van Au, V. & Xu, R. (2021). *Semilinear Caputo time-fractional pseudo-parabolic equations*, Commun. Pure Appl. Anal., *20*(2), 583-621.

[16] Huy Tuan, N. (2022). *On some inverse problem for Bi-Parablloic equation with observed data in L^p spaces*, Opuscula Math., *42*(2), 305–335.

[17] Huy Tuan, N. & Caraballo, T. (2021). *On initial and terminal value problems for fractional nonclassical diffusion equations*, Proc. Amer. Math. Soc., 149(1), 143–161.

[18] Viet Tri, V. & Rezapour, S. (2021). *Eigenvalue Intervals of Multivalued Operator and its Application for a Multipoint Boundary Value Problem*, Bulletin of the Iranian Mathematical Society, *47*(4), 1301-1314.

[19] Viet Tri, V. & Karapinar, E. (2020). *A Fixed Point Theorem and an Application for the Cauchy Problem in the Scale of Banach Spaces*, Filomat, *34*(13), 4387-4398.