High-order iterative scheme to the Robin problem for a nonlinear wave equation with viscoelastic term

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ABSTRACT
The report deals with the Robin problem for a nonlinear wave equation with viscoelastic term. Under some suitable conditions, we establish a high-order iterative scheme and then prove that the scheme converges to the weak solution of the original problem along with the error estimate. This result extends the result in [9].

Keywords: Faedo-Galerkin method, High-order iterative scheme, Nonlinear wave equation, Local existence.

1 Introduction
This report is devoted to study the Robin problem for a nonlinear wave equation with viscoelastic term as follows

\[
\begin{aligned}
&u_{tt} - u_{xx} + \lambda(x, t, u) |u_t|^{q-2} u_t + \int_0^t g(t-s) u_{xx}(x, s) ds = f(x, t, u), \\
&u_x(0, t) - u(0, t) = u_x(1, t) + u(1, t) = 0, \\
u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x),
\end{aligned}
\]

where \(q \geq 2\) is a given constant and \(\lambda, f, g, \tilde{u}_0, \tilde{u}_1\) are given functions with \(\lambda(x, t, u) \geq \lambda_* > 0\).

Equation (1.1) usually arises within frameworks of mathematical models in engineering and physical sciences. The left-hand integral of equation (1.1) is called viscoelastic term.

When \(\lambda(x, t, u) \equiv a, f \equiv b |u|^{p-2} u\), equation (1.1) becomes the following nonlinear wave equation

\[
u_{tt} - \Delta u + a |u_t|^{q-2} u_t = b |u|^{p-2} u,
\]

where \(a, b > 0\) and \(p, q \geq 2\). This equation has been widely studied and obtained many interesting results such as the global existence, exponential decay and finite-time blow-up of solutions (see [1], [2], [4], [10], [12]).
When $\lambda(x, t, u) \equiv 1$ and $f \equiv b|u|^{p-2}u$, equation (1.1) is reduced to the viscoelastic wave equation of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x, s)ds + |u_t|^{q-2}u_t = |u|^{p-2}u,$$

this form was considered by Messaoudi in [6], where the author proved a finite-time blow-up result for solutions with negative initial energy if $p > q$ and a global existence result for $q \geq p$.

In this paper, we associate with equation (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{align*}
  u_0 &\equiv 0, \\
  u'_m - \Delta u_m + \lambda(x, t, u_m)|u'_m|^{q-2}u'_m + \int_0^t g(t-s)\Delta u_m(s)ds \\
  &= \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x, t, u_{m-1}) (u_m - u_{m-1})^i, \quad 0 < x < 1, \quad 0 < t < T, \\
  u_{mx}(0, t) - u_{m}(0, t) &= u_{mx}(1, t) + u_m(1, t) = 0, \\
  u_m(x, 0) &= \hat{u}_0(x), \quad u_{mx}(x, 0) = \hat{u}_1(x), \quad m = 1, 2, \ldots .
\end{align*}$$

If $\lambda \in C^1([0, 1] \times [0, T^*] \times \mathbb{R})$, $\lambda(x, t, u) \geq \lambda_*, g \in H^1(0, T^*)$, $f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R})$ and some other conditions, we prove that the sequence $\{u_m\}$ converges at the $N$-order rate to the unique weak solution of Prob. (1.1), it means that

$$\|u_m - u\|_X \leq C \|u_{m-1} - u\|_X^N,$$

for some $C > 0$, where $X$ is a suitable space. The scheme (1.4) is called the high-order iterative scheme or the $N$-order iterative scheme. We note more that the high-order iterative schemes as above were also used to obtain the existence of solutions in the previous papers, for example, see [7], [8], [9], [11].

This paper consists of four sections. Section 2 is devoted to the presentation of preliminaries. In Section 3, by using the Faedo-Galerkin approximation method and the arguments of compactness, we prove Theorem 3.1 to get the high-order iterative scheme (1.4). Finally, in Section 4, we prove Theorem 4.1 to obtain the convergence of the high-order iterative scheme (1.4) and then, the unique existence of a weak solution of Prob. (1.1) follows. The result obtained here is a generalization of the results of [9] and based on the ideas about recurrence relations as in [7], [8], [9], [11].

## 2 Preliminaries

Put $\Omega = (0, 1)$. We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in $L^2$ or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in $L^2$ and $\|\cdot\|_X$ is the norm in the Banach space $X$. We call $X'$ the dual space of $X$. We denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0, T) \to X$ measurable, such that $\|u\|_{L^p(0, T; X)} < +\infty$, with

$$\|u\|_{L^p(0, T; X)} = \begin{cases} \\
  \left( \int_0^T \|u(t)\|^p_X dt \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\
  \text{ess sup}_{0 < t < T}\|u(t)\|_X, & \text{if } p = \infty.
\end{cases}$$

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We write \( u(t), u'(t) = u_t(t) = \dot{u}(t), u''(t) = u_{tt}(t) = \ddot{u}(t), u_x(t) = \nabla u(t), u_{xx}(t) = \Delta u(t), \) to denote \( u(x, t), \frac{\partial u}{\partial t}(x, t), \frac{\partial^2 u}{\partial t^2}(x, t), \frac{\partial u}{\partial x}(x, t), \frac{\partial^2 u}{\partial x^2}(x, t), \) respectively. With \( f \in C^k([0, 1] \times \mathbb{R}_+ \times \mathbb{R}), \)
\( f = f(x, t, u), \) we put \( D_1f = \frac{\partial f}{\partial x}, D_2f = \frac{\partial f}{\partial t}, D_3f = \frac{\partial f}{\partial u} \) and \( D^\alpha f = D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}f; \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq k, D^{(0,0,0)}f = f. \)

On \( H^1, \) we shall use the following norm
\[
\|v\|_{H^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}.
\]

We also define the following bilinear form and the other norms on \( H^1 \)
\[
a(u, v) = \int_0^1 u_x(x)v_x(x)dx + u(0)v(0) + u(1)v(1), \forall u, v \in H^1, \tag{2.1}
\]

\[
\|v\|_a = \sqrt{a(v, v)}, \forall v \in H^1,
\] and
\[
\|v\|_i = \left( v^2(i) + \int_0^1 v_x^2(x)dx \right)^{1/2}, i = 0, 1. \tag{2.3}
\]

On \( H^1, \) three norms \( \|v\|_{H^1}, \|v\|_a \) and \( \|v\|_i \) are equivalent norms.

We now have the following lemmas, the proofs of which are straightforward so we omit the details.

**Lemma 2.1.** The imbedding \( H^1 \hookrightarrow C^0(\bar{\Omega}) \) is compact and

\[
\begin{align*}
(i) \quad & \|v\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|v\|_{H^1}, \\
(ii) \quad & \|v\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|v\|_i, \\
(iii) \quad & \frac{1}{\sqrt{3}} \|v\|_{H^1} \leq \|v\|_i \leq \sqrt{3} \|v\|_{H^1},
\end{align*}
\]

for all \( v \in H^1, i = 0, 1. \)

**Lemma 2.2.** The symmetric bilinear form \( a(\cdot, \cdot) \) defined by (2.1) is continuous on \( H^1 \times H^1 \) and coercive on \( H^1, \) i.e.,

\[
\begin{align*}
(i) \quad & |a(u, v)| \leq 5 \|u\|_{H^1} \|v\|_{H^1}, \text{ for all } u, v \in H^1, \\
(ii) \quad & a(u, u) \geq \frac{1}{3} \|u\|^2_{H^1}, \text{ for all } u \in H^1.
\end{align*}
\]

3 Main results

3.1 A high-order iterative scheme

In this section, we shall establish a high-order iterative scheme in order to obtain the existence of a weak solution for Prob. (1.1). Let us note here that the weak solution \( u \) of Prob. (1.1) will be obtained in Section 4 (Theorem 4.1) in the following manner:

Find \( u \in L^\infty(0, T; H^2) \) such that \( u' \in L^\infty(0, T; H^1), u'' \in L^\infty(0, T; L^2) \) and \( u \) satisfies the following variational problem and the initial conditions

\[
\begin{align*}
\langle u''(t), w \rangle + a(u(t), w) + \langle \lambda(t, u(t)) \rangle u'(t) |u'(t)|^{q-2} u'(t), w \\
= \int_0^t g(t-s)a(u(s), w)ds + \langle f(x, t, u), w \rangle, \forall w \in H^1,
\end{align*}
\]

\[
u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1,
\]

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where $a(\cdot, \cdot)$ is the symmetric bilinear form on $H^1$ defined by (2.1).

Let $T^* > 0$, we make the following assumptions:

\begin{itemize}
  \item [(H1)] $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$;
  \item [(H2)] $g \in H^1(0, T^*)$;
  \item [(H3)] $\lambda \in C^1([0, 1] \times [0, T^*] \times \mathbb{R})$, and there exists a positive constant $\lambda_*$ such that $\lambda(x, t, u) \geq \lambda_* > 0$, $\forall (x, t, u) \in [0, 1] \times [0, T^*] \times \mathbb{R}$;
  \item [(H4)] $f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R})$ such that
    \begin{itemize}
      \item [(i)] $D_1^i f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R})$, $1 \leq i \leq N$,
      \item [(ii)] $D_1 D_2^i f \in C^0([0, 1] \times \mathbb{R}_+ \times \mathbb{R})$, $0 \leq i \leq N - 1$.
    \end{itemize}
\end{itemize}

Fix $T^* > 0$. For each $T \in (0, T^*)$ and $M > 0$, we put

\begin{equation}
\begin{aligned}
W(M, T) &= \{ v \in L^\infty(0, T; H^2) : v' \in L^\infty(0, T; H^1), v'' \in L^2(Q_T), \\
& \quad \text{with } \| v \|_{L^\infty(0, T; H^2)}, \| v' \|_{L^\infty(0, T; H^1)}, \| v'' \|_{L^2(Q_T)} \leq M \}, \\
W_1(M, T) &= \{ v \in W(M, T) : v'' \in L^\infty(0, T; L^2) \}.
\end{aligned}
\end{equation}

Now, we construct the following recurrent sequence $\{ u_m \}$:

The first term is chosen as $u_0 \equiv 0$, suppose that

\begin{equation}
\begin{aligned}
   u_{m-1} &\in W_1(M, T), \quad (3.3)
\end{aligned}
\end{equation}

we find $u_m \in W_1(M, T)$ ($m \geq 1$) satisfying the nonlinear variational problem

\begin{equation}
\begin{aligned}
   & \begin{cases}
   \langle u_m'(t), w \rangle + a(u_m(t), w) + \langle \lambda(t, u_m(t)) \rangle u_m'(t) \| u_m'(t) \|^{q-2} u_m'(t), w \rangle \\
   \quad = \int_0^t g(t-s) a(u_m(s), w) ds + \langle F_m(t), w \rangle, \forall w \in H^1,
   \end{cases} \\
   & u_m(0) = \tilde{u}_0, u_m'(0) = \tilde{u}_1,
\end{aligned}
\end{equation}

in which

\begin{equation}
F_m(x, t) = \sum_{i=0}^{N-1} \frac{1}{i!} D^i_2 f(x, t, u_{m-1}) (u_m - u_{m-1})^i.
\end{equation}

Then we have the following theorem.

**Theorem 3.1.** Let $(H_1) - (H_4)$ hold. Then there exist a constant $M > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$ and a constant $T > 0$ depending on $\tilde{u}_0$, $\tilde{u}_1$, $g$, $f$, $q$ and $\lambda$ such that, for $u_0 \equiv 0$, there exists a recurrent sequence $\{ u_m \} \subset W_1(M, T)$ defined by (3.4)-(3.5).

**Proof.** The proof is based on the Faedo - Galerkin approximation method introduced by Lions [5], the arguments of compactness, together with the same evaluation techniques as in [9]. \qed

### 3.2 Convergence and error estimate of the scheme

This section is devoted to prove the $N$-order convergence of the sequence $\{ u_m \}$ established in Theorem 3.1 to the weak solution of Prob. (1.1). First, we denote

\begin{equation}
W_1(T) = C([0, T] ; H^1) \cap C^1([0, T] ; L^2),
\end{equation}

it is clear to see that $W_1(T)$ is a Banach space with respect to the norm

\begin{equation}
\| v \|_{W_1(T)} = \| v \|_{C([0, T] ; H^1)} + \| v' \|_{C^0([0, T] ; L^2)}.
\end{equation}

Then we have the following theorem.
Theorem 3.2. Let \((H_1) - (H_4)\) hold. Then, there exist constants \(M > 0\) and \(T > 0\) defined as in Theorem 3.1 such that

(i)\ Prob. \((1.1)\) has a unique weak solution \(u \in W_1(M, T)\) and the sequence \(\{u_m\}\) defined by \((3.4)-(3.5)\) converges at a rate of order \(N\) to the solution \(u\) strongly in the space \(W_1(T)\), in the sense

\[
\|u_m - u\|_{W_1(T)} \leq C \|u_{m-1} - u\|^N_{W_1(T)},
\]

for all \(m \geq 1\), where \(C\) is a suitable constant.

(ii) Furthermore, the following estimate is fulfilled

\[
\|u_m - u\|_{W_1(T)} \leq C_T (\gamma_T)^{N^m}, \text{ for all } m \in \mathbb{N},
\]

where \(C_T\) and \(0 < \gamma_T < 1\) are the constants depending only on \(T\).

Proof. (i) Existence of a solution. We shall prove that \(\{u_m\}\) is a Cauchy sequence in \(W_1(T)\). Indeed, we put \(v_m = u_{m+1} - u_m\). Then \(v_m\) satisfies the variational problem

\[
\begin{aligned}
&\int_0^t \left< \lambda(s, u_{m+1}(s)) \left[ |u_{m+1}'(s)|^{q-2} u_{m+1}'(s) - |u_m'(s)|^{q-2} u_m'(s) \right], v_m(s) \right> ds \\
&+ \int_0^t g(t - s) a(v_m(s), w) ds + \langle F_{m+1}(t) - F_m(t), w \rangle, \forall w \in H^1,
\end{aligned}
\]

Taking \(w = v_m'\) in \((3.10)\), after integrating in \(t\), and noting that

\[
-2\int_0^t \left< \lambda(s, u_{m+1}(s)) \left[ |u_{m+1}'(s)|^{q-2} u_{m+1}'(s) - |u_m'(s)|^{q-2} u_m'(s) \right], v_m'(s) \right> ds \leq 0,
\]

we get

\[
X_m(t) \leq -2\int_0^t \left< [\lambda(s, u_{m+1}(s)) - \lambda(s, u_m(s))] |u_m'(s)|^{q-2} u_m'(s), v_m'(s) \right> ds + 2\int_0^t g(t - s) a(v_m(s), v_m(s)) ds
\]

\[
+ 2\int_0^t ds \int_0^s g'(s - \tau) a(v_m(\tau), v_m(s)) d\tau
\]

\[
- 2\int_0^t ds \int_0^s \langle F_{m+1}(s) - F_m(s), v_m'(s) \rangle ds,
\]

\[
\equiv \sum_{k=1}^5 J_k,
\]

with

\[
X_m(t) = \|v_m'(t)\|^2 + \|v_m(t)\|^2_{a}.
\]

We denote the constants \(K_M(f), \overline{K}_M(\lambda), \) as follows

\[
\begin{cases}
K_M(f) = \|f\|_{C^0(\Omega_M)} + \sum_{i=1}^N \|D_i^3f\|_{C^0(\Omega_M)} + \sum_{i=1}^{N-1} \|D_1 D_i^3 f\|_{C^0(\Omega_M)}, \\
\|f\|_{C^0(\Omega_M)} = \sup_{(x,t,u)\in\Omega_M} |f(x, t, u)|, \\
\overline{K}_M(\lambda) = \|D_3 \lambda\|_{C^0(\Omega_M)}, \\
\Omega_M = [0, 1] \times [0, T^*] \times [-\sqrt{2M}, \sqrt{2M}].
\end{cases}
\]
Next, we need to estimate the integrals on the right side of (3.11) as follows. First, it is not difficult to estimate terms $J_1$, $J_2$, $J_3$ and $J_4$ as follows:

\[ J_1 = -2 \int_0^t \langle [\lambda(s, u_{m+1}(s)) - \lambda(s, u_m(s))] u_m'(s), v_{m}'(s) \rangle ds \]  
\[ \leq 2K_M(\lambda)M^{q-1} \int_0^t \|v_m(s)\| \|v_m'(s)\| ds \leq K_M(\lambda)M^{q-1} \int_0^t X_m(s) ds; \]

\[ J_2 = 2 \int_0^t g(t - \tau)a(v_m(\tau), v_m(t)) d\tau \leq \frac{1}{2} X_m(t) + 2 \|g\|_{L^2(0, T^*)}^2 \int_0^t X_m(s) ds; \]

\[ J_3 = -2 \int_0^t \langle g(0) a(v_m(s), v_m(s)) ds \leq 2 |g(0)| \int_0^t X_m(s) ds; \]

\[ J_4 = -2 \int_0^t ds \int_0^s g(s - \tau)a(v_m(\tau), v_m(s)) d\tau \leq 2\sqrt{T} \|g\|_{L^2(0, T^*)} \int_0^t X_m(s) ds. \]

Next, using Taylor’s expansion of the function $f(x, t, u_m) = f(x, t, u_{m-1} + v_{m-1})$ around the point $u_{m-1}$ up to order $N$, we obtain

\[ f(x, t, u_m) - f(x, t, u_{m-1}) = \sum_{i=1}^{N-1} \frac{1}{i!} D^i f(x, t, u_{m-1}) v^i_{m-1} + \frac{1}{N!} D^N f(x, t, \tilde{\theta}_m) v^N_{m-1}, \]  

where $\tilde{\theta}_m = \tilde{\theta}(x, t) = u_m + \theta_1 v_{m-1}$, $0 < \theta_1 < 1$.

Hence, it follows from (3.5) and (3.15) that

\[ F_{m+1}(x, t) - F_m(x, t) = \sum_{i=1}^{N-1} \frac{1}{i!} D^i f(x, t, u_m) v^i_m + \frac{1}{N!} D^N f(x, t, \tilde{\theta}_m) v^N_{m-1}. \]  

Therefore, we have

\[ \|F_{m+1}(t) - F_m(t)\| \leq K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} (\sqrt{2} \|v_m(t)\|_{H^1})^i + \frac{1}{N!} K_M(f) (\sqrt{2} \|v_{m-1}(t)\|_{H^1})^N \]  
\[ \leq \beta_T^{(1)} \sqrt{X_m(t)} + \beta_T^{(2)} \|v_{m-1}\|_{W^1(T)}^N, \]  

where $\beta_T^{(1)} = \sqrt{6} K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} (\sqrt{2}M)^{i-1}$, $\beta_T^{(2)} = \frac{\sqrt{2}N}{N!} K_M(f)$.

It implies that

\[ J_5 = 2 \int_0^t \langle F_{m+1}(s) - F_m(s), v_m'(s) \rangle ds \]  
\[ \leq 2 \int_0^t \|F_{m+1}(s) - F_m(s)\| \|v_m'(s)\| ds \]
\[ \leq 2 \int_0^t \left( \beta_T^{(1)} \sqrt{X_m(s)} + \beta_T^{(2)} \|v_{m-1}\|_{W^1(T)}^N \right) \sqrt{X_m(s)} ds \]
\[ \leq 2\beta_T^{(1)} \int_0^t X_m(s) ds + 2\beta_T^{(2)} \|v_{m-1}\|_{W^1(T)}^N \int_0^t \sqrt{X_m(s)} ds \]
\[ \leq 2\beta_T^{(1)} \int_0^t X_m(s) ds + T\beta_T^{(2)} \|v_{m-1}\|_{W^1(T)}^{2N} + \beta_T^{(2)} \int_0^t X_m(s) ds. \]

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Combining (3.11), (3.14) and (3.18), we obtain

\[ X_m(t) \leq 2T \beta_T^{(2)} \|v_{m-1}\|_{W_1(T)}^{2N} + \beta_T^{(3)} \int_0^t X_m(s) ds, \]  

(3.19)

where

\[ \beta_T^{(3)} = 2 \left[ K_M(\lambda)M^{q-1} + 2 \left( |g(0)| + \|g\|_{L^2(0,T^*)}^2 + \sqrt{T^*} \|g'\|_{L^2(0,T^*)} + \beta_T^{(1)} + \beta_T^{(2)} \right) \right]. \]

By using Gronwall’s lemma, (3.19) gives

\[ \|v_m\|_{W_1(T)} \leq \mu_T \|v_{m-1}\|_{W_1(T)}^N, \]

(3.20)

with \( \mu_T = (1 + \sqrt{3}) \sqrt{2T \beta_T^{(2)} \exp(T \beta_T^{(3)})}. \)

Choosing \( T > 0 \) small enough such that \( \gamma_T = M \mu_T \frac{1}{T} < 1 \), it follows from (3.20) that

\[ \|u_m - u_{m+p}\|_{W_1(T)} \leq (1 - \gamma_T)^{-1} T^{\frac{1}{N+1}} (\gamma_T)^N, \]

for all \( m \) and \( p \in \mathbb{N} \).

(3.21)

Hence, \( \{u_m\} \) is a Cauchy sequence in \( W_1(T) \). Thus, there exists \( u \in W_1(T) \) such that

\[ u_m \to u \text{ strongly in } W_1(T). \]

(3.22)

Note that \( u_m \in W_1(M, T) \), then there exists a subsequence \( \{u_{m_j}\} \) of \( \{u_m\} \) such that

\[
\begin{align*}
\{u_{m_j}\} & \to u \quad \text{in } L^\infty(0, T; H^2) \text{ weakly*}, \\
u'_{m_j} & \to u' \quad \text{in } L^\infty(0, T; H^1) \text{ weakly*}, \\
u''_{m_j} & \to u'' \quad \text{in } L^2(Q_T) \text{ weakly}, \\
u & \in W(M, T).
\end{align*}
\]

Moreover, by (3.22) and the inequalities

\[ \sup_{0 \leq t \leq T} \|\lambda(t, u_m(t)) - \lambda(t, u(t))\| \leq K_M(\lambda) \|u_m - u\|_{W_1(T)}, \]

(3.24)

\[ \left\|\left|u_{m}^{\prime q-2} u_m' - u' q-2 u' \right\|_{C^0([0, T]; L^2)} \right\| \leq (q - 1) \left( \sqrt{2M} \right)^{q-2} \|u_m - u\|_{W_1(T)}, \]

we have

\[ \lambda(\cdot, t, u_m(t)) \to \lambda(\cdot, t, u(t)) \text{ strongly in } C^0([0, T]; L^2), \]

(3.25)

\[ |u_m'|^{q-2} u_m' \to |u'|^{q-2} u' \text{ strongly in } C^0([0, T]; L^2). \]

On the other hand

\[ \|F_m(\cdot, t) - F(\cdot, t, u(t))\| \]

(3.26)

\[ \leq \|f(\cdot, t, u_{m-1}(t)) - f(\cdot, t, u(t))\| + \left\| \sum_{i=1}^{N-1} \frac{1}{i!} D_i f(\cdot, t, u_{m-1})(u_m - u_{m-1})^i \right\| \]

\[ \leq K_M(\lambda) \left\| u_{m-1} - u \right\|_{W_1(T)} + \sum_{i=1}^{N-1} \frac{1}{i!} \|u_m - u_{m-1}\|_{W_1(T)}^i. \]
Therefore, it implies from (3.22) and (3.25) that
\[ F_m(t) \to f(\cdot, t, u(t)) \] strongly in \( C^0([0, T]; L^2) \).

Finally, passing to limit in (3.4) and (3.5) as \( m = m_j \to \infty \), there exists \( u \in W(M, T) \) satisfying the equation
\[
\langle u''(t), w \rangle + a(u(t), w) + \langle \lambda(t, u(t)) |u'(t)|^{q-2} u'(t), w \rangle
= \int_0^t g(t-s)a(u(s), w)ds + \langle f(\cdot, t, u(t)), w \rangle,
\]
for all \( w \in H^1 \) and the initial condition
\[
 u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1. \tag{3.29}
\]

On the other hand, it follows from (3.23)\textsubscript{4} and (3.28) that
\[
 u'' = \Delta u - \lambda(x, t, u) |u'|^{q-2} u' + \int_0^t g(t-s)\Delta u(s)ds + f(x, t, u) \in L^\infty(0, T; L^2), \tag{3.30}
\]
hence, \( u \in W_1(M, T) \).

**Uniqueness.** Let \( u_1, u_2 \in W_1(M, T) \) be two weak solutions of Prob. (1.1). Then \( \bar{u} = u_1 - u_2 \) satisfies the variational problem
\[
\begin{cases}
\langle \bar{u}''(t), w \rangle + a(\bar{u}(t), w) \\
= -\langle \lambda(t, u_1(t)) (|u_1'(t)|^{q-2} u_1'(t) - |u_2'(t)|^{q-2} u_2'(t)), w \rangle \\
-\langle (\lambda(t, u_1(t)) - \lambda(t, u_2(t))) |u_2'(t)|^{q-2} u_2'(t), w \rangle \\
+ \int_0^t g(t-s)a(\bar{u}(s), w)ds + \langle f(x, t, u_1) - f(x, t, u_2), w \rangle, \quad \forall w \in H^1,
\end{cases}
\]
\[
\bar{u}(0) = \bar{u}'(0) = 0. \tag{3.31}
\]

We take \( w = \bar{u}'(t) \) in (3.31)\textsubscript{1} and integrate in \( t \) to get
\[
\rho(t) = \|\bar{u}'(t)\|^2 + \|\bar{u}(t)\|^2 \leq 2 \int_0^t \langle (\lambda(s, u_1(s)) - \lambda(s, u_2(s))) |u_2'(s)|^{q-2} u_2'(s), \bar{u}'(s) \rangle ds
\]
\[
+ 2 \int_0^t g(t-\tau)a(\bar{u}(\tau), \bar{u}(\tau)) d\tau - 2 \int_0^t g(0) a(\bar{u}(\tau), \bar{u}(\tau)) ds
\]
\[
- 2 \int_0^t ds \int_0^s g'(s-\tau)a(\bar{u}(\tau), \bar{u}(\tau)) d\tau
\]
\[
+ 2 \int_0^t \langle f(x, s, u_1(s)) - f(x, s, u_2(s)), \bar{u}'(s) \rangle ds
\]
\[
\equiv \sum_{k=1}^4 j_k, \tag{3.33}
\]

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We estimate the integrals $\bar{J}_k$, $k = 1, 5$ as follows.

\[
\begin{align*}
\bar{J}_1 &= -2\int_0^t \left( (\lambda(s, u_1(s)) - \lambda(s, u_2(s))) |u'_2(s)| + u'_2(s), \bar{u}'(s) \right) ds \\
&\leq 2\bar{K}_M(\lambda)M^{q-1}\int_0^t \|\bar{u}(s)\| \|\bar{u}'(s)\| ds \\
&\leq \bar{K}_M(\lambda)M^{q-1}\int_0^t \rho(s) ds;
\end{align*}
\]

\[
\begin{align*}
\bar{J}_2 &= 2\int_0^t g(t - \tau) a(\bar{u}(\tau), \bar{u}(t)) d\tau \\
&\leq \frac{1}{2} \rho(t) + 2 \|g\|_{L^2(0,T^*)} \int_0^t \rho(s) ds;
\end{align*}
\]

\[
\begin{align*}
\bar{J}_3 &= -2\int_0^t g(0) a(\bar{u}(s), \bar{u}(s)) ds \\
&\leq 2 |g(0)| \int_0^t \rho(s) ds;
\end{align*}
\]

\[
\begin{align*}
\bar{J}_4 &= -2\int_0^t ds \int_0^s g'(s - \tau) a(\bar{u}(\tau), \bar{u}(s)) d\tau \\
&\leq 2\sqrt{T^*} \|g'\|_{L^2(0,T^*)} \int_0^t \rho(s) ds;
\end{align*}
\]

\[
\begin{align*}
\bar{J}_5 &= 2\int_0^t \left( f(x, s, u_1(s)) - f(x, s, u_2(s)), \bar{u}'(s) \right) ds \\
&\leq 2\sqrt{6}K_M(f) \int_0^t \rho(s) ds.
\end{align*}
\]

We deduce from (3.32) and (3.34), that

\[
\rho(t) = \|\bar{u}'(t)\|^2 + \|\bar{u}(t)\|_a^2 \leq k_T \int_0^t \rho(s) ds,
\]

where

\[
k_T = 2 \left[ \bar{K}_M(\lambda)M^{q-1} + 2 \left( \|g\|_{L^2(0,T^*)}^2 + |g(0)| + \sqrt{T^*} \|g'\|_{L^2(0,T^*)} + \sqrt{6}K_M(f) \right) \right].
\]

Using Gronwall’s Lemma, it follows that $\rho(t) = \|\bar{u}'(t)\|^2 + \|\bar{u}(t)\|_a^2 \equiv 0$, i.e., $\bar{u} = u_1 - u_2 = 0$. Therefore, $u \in W_4(M, T)$ is an unique local weak solution of Prob. (1.1).

(ii) Passing to the limit in (3.21) as $p \to \infty$ for fixed $m$, we get (3.9).

By the similar argument, (3.8) follows. Theorem 3.2 is proved completely.

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**References**


